Instantons in a ferromagnetic spin chain with biaxial anisotropy

G.-P. Zheng,[1] J.-Q. Liang,[2] and W. M. Liu[3]
[1]*Department of Physics, Henan Normal University, Xinxiang, Henan 453007, China*
[2]*Department of Physics and Institute of Theoretical Physics, Shanxi University, Taiyuan, Shanxi 030006, China*
[3]*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China*
(Received 23 June 2007; revised manuscript received 6 November 2008; published 13 January 2009)

Instanton configurations of 1+1 dimensions in a ferromagnetic biaxial-anisotropy-spin chain are obtained explicitly in the strong-anisotropy limit, and macroscopic quantum coherent tunneling of domain walls is illustrated in terms of the instantons. The tunneling amplitude for an instanton configuration is obtained and the result shows that it is possible to have a finite-energy splitting of the degenerate orientations of the magnetization for a finite-length spin chain at finite temperature.

DOI: 10.1103/PhysRevB.79.014415 PACS number(s): 75.45.+j, 11.27.+d, 75.10.Pq

I. INTRODUCTION

Quantum tunneling has attracted considerable interest because of its wide applications in areas ranging from condensed matter to high-energy physics and cosmology. The instanton method[1,2] is a powerful tool for the calculation of tunneling effects. Instantons, which are classical Euclidean configurations connecting degenerate minima of the Euclidean potential, are seen to be just the classical trajectories of pseudoparticles existing in the potential barrier region and not actual field configurations of real time. The tunneling amplitude between two degenerate states is, however, dominated by the instanton configurations in the semiclassical expansion of the functional integral. In the context of quantum mechanics the instantons, which are obtained from 1+0 dimensional Euclidean equation of motion, have been well studied for various potentials such as double well[3] and sine-Gordon.[4]

Recently the quantum tunneling of magnetization vector in single domain ferromagnetic grain and Néel vector in single domain antiferromagnetic grain has become an active research field[5,6] both theoretically and experimentally. With the help of spin-coherent-state path integrals, the spin systems can be converted to typical potential tunneling models of 1+0 dimension. Although quantum tunneling in the context of field theory, for instance, the tunneling of domain walls,[7–11] has been investigated, the explicit multidimensional (time-space) instanton configurations have not been studied extensively.

We in the present paper give some explicit instanton configurations of 1+1 dimensions and illustrate the quantum coherent tunneling of domain walls in a continuous ferromagnetic spin chain of biaxial anisotropy. In Ref. 12, the authors studied the macroscopic quantum tunneling in an antiferromagnetic spin chain and given the pure-quantum-tunneling configurations for the sine-Gordon model. But in this paper, we investigate the finite-temperature effects, namely, the thermally assisted quantum tunneling and obtain the periodic instanton configurations.

The paper is organized as follows. In Sec. II, we introduce the model of ferromagnetic spin chain with biaxial anisotropy and the equation of motion of the magnetization vector is reduced to the 1+1-dimensional sine-Gordon field equation in strong-anisotropy limit. In Sec. III, an instanton configuration and corresponding tunneling amplitude are obtained analytically, and the tunneling process of the domain walls is illustrated. In Sec. IV, we give the breatherlike instanton. Periodic instanton–anti-instanton pair configurations are shown explicitly in Sec. V. Finally, we summarize our results in Sec. VI.

II. MODEL

We consider a spin chain consisting of molecular ferromagnets with biaxial anisotropy, for instance, the molecule Fe_8. The model Hamiltonian is seen to be

$$\hat{H} = -J\sum_{\langle i,j\rangle} \hat{S}_i \cdot \hat{S}_j + K_1\sum_i (\hat{S}_i^z)^2 - K_2\sum_i (\hat{S}_i^x)^2, \tag{1}$$

where $J>0$ is the exchange parameter, and K_1 and K_2 are the anisotropy parameters with $K_1 > K_2 > 0$. The giant spin of the molecular magnet at ith site is described by the spin operator $\hat{S}_i$. The bracket $\langle i,j\rangle$ means the sum over the nearest neighbors only.

The ferromagnet at each lattice site has a XY easy plane with a X easy axis. As a consequence of the square term $(\hat{S}_i^x)^2$, the two equilibrium orientations of the magnetization along the $\pm X$ directions are degenerate. Thus this model possesses instanton configurations, which are responsible for the quantum tunneling[3,4,13] of the domain walls.

The Heisenberg equation of motion for the spin operator of the ferromagnet at the kth site is $\frac{d}{dt}\hat{S}_k = \frac{1}{i\hbar}[\hat{S}_k, \hat{H}]$. At low temperatures, the operator of the giant spin can be treated as[14] a classical vector $\hat{S}_k \to \vec{S}_k$. Taking the continuous limit, $\vec{S}_k \to \vec{S}(x,t)$, we have the differential equations of the spin vector in the spherical coordinates, $\vec{S} = (S^x, S^y, S^z) = S(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, with $S = s(s+1)\hbar$, where s is the spin quantum number.[15] For strong anisotropy, i.e., $K_1 \gg K_2$, the spin vectors are forced to rotate in the XY plane and we may assume that $\theta = \pi/2 - \delta$, where δ denotes a small perturbation angle. Expanding θ to the first order of δ and assuming slow spatial variation in angles δ and φ, the equation of motion then reads

$$\frac{\partial \delta}{\partial t} = 2JS\frac{\partial^2 \varphi}{\partial x^2} - K_2 S \sin(2\varphi),$$

$$\frac{\partial \varphi}{\partial t} = 2K_1 S\delta, \tag{2}$$

where we have assumed that the chain is along X direction and x denotes the dimensionless spatial coordinate measured in the unit of the lattice constant. As a result of the above approximation, one has

$$2JS\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{2K_1 S}\frac{\partial^2 \varphi}{\partial t^2} = K_2 S \sin(2\varphi). \tag{3}$$

A Lagrangian density that yields the equation of motion can be written as[15]

$$\pounds = 2\hbar JS\left[\frac{1}{2c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 - V(\varphi)\right], \tag{4}$$

with $V(\varphi)=\frac{1}{g^2}(1-\cos 2\varphi)$, where $c=2S\sqrt{JK_1}$ is the speed of the spin wave[12] and $g=\sqrt{4J/K_2}$ is a dimensionless parameter. The canonical momentum density is defined by $\Pi = \partial\pounds / \partial(\frac{\partial\varphi}{\partial t}) = \frac{2\hbar JS}{c^2}(\frac{\partial\varphi}{\partial t})$, and we have the Hamiltonian density $H=2\hbar JS[\frac{1}{2c^2}(\frac{\partial\varphi}{\partial t})^2+\frac{1}{2}(\frac{\partial\varphi}{\partial x})^2+V(\varphi)]$.

III. INSTANTONS AND TUNNELING OF DOMAIN WALLS

The instanton is a solution of Euclidean field equation of motion

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{c^2\,\partial \tau^2} = \frac{2}{g^2}\sin(2\varphi), \tag{5}$$

for the Euclidean Lagrangian density $\pounds_e=2\hbar JS[\frac{1}{2c^2}(\frac{\partial\varphi}{\partial\tau})^2+\frac{1}{2}(\frac{\partial\varphi}{\partial x})^2+V(\varphi)]$, where $\tau=it$ is imaginary time. The instanton configuration is found as (see the Appendix)

$$\cos[\varphi(x,\tau)] = k\ \mathrm{sn}\left[\pm\frac{2(x-v\tau-x_0)}{g\sqrt{1+(v/c)^2}},k\right], \tag{6}$$

where v is the velocity of the instanton and sn is Jacobian elliptic function[16] with modulus $k=\sqrt{1-g^2E_{cl}[1+(v/c)^2]/2}$, where the integration constant E_{cl} plays the role of dimensionless energy density. The integration constant x_0 is the center position of the instanton at $\tau=0$. For simplicity, we take it to be zero.

We demand that $\varphi(x,\tau)$ be periodic with period L along the spin chain, $\varphi(x,\tau)=\varphi(x+L,\tau)$, where L denotes the dimensionless length of the chain measured in unit of lattice constant. This leads to the condition

$$\frac{2L}{g\sqrt{[1+(v/c)^2]}} = 4mK(k), \tag{7}$$

where $m=1,2,3,\ldots$ are positive integer and $K(k)$ denotes the complete elliptic integral of the first kind.[16]

In statistical mechanics, the mathematical analogy of the density matrix[17] and the transition amplitude in imaginary time offer the period of the periodic instanton at given temperature T[6],

$$p = \frac{\hbar}{K_B T} = \frac{2K(k)g\sqrt{[1+(v/c)^2]}}{v}, \tag{8}$$

where $\hbar$ and K_B are the Planck and the Boltzmann constants, respectively. The parameter k is to be determined from the above two equations. We also have

$$v = \frac{K_B TL}{m\hbar}, \tag{9}$$

namely, the velocity of the instantons are quantized due to the periodic boundary condition.

Inserting Eq. (9) into Eq. (8), we obtain the dependence relation of k on temperature T

$$K(k) = \frac{L}{2gm}\frac{1}{\sqrt{\left[1+\left(\frac{K_B TL}{m\hbar c}\right)^2\right]}}. \tag{10}$$

Then we have the T-dependent energy density $E_{cl}(T) = \frac{2(1-k^2)}{g^2[1+(v/c)^2]}$. Recalling that $K(k)\geq \pi/2$, we get the restriction for our solution

$$T \leq \frac{m\hbar c}{K_B L}\sqrt{\left(\frac{L}{mg\pi}\right)^2 - 1}. \tag{11}$$

As the temperature is increased, the number of instanton and anti-instantons increased and the dilute gas approximation[2] breaks down. For $k\to 0$, we have the maximal temperature at which the periodic instanton and anti-instantons condense and become an instanton solid.[6] In this limit, the energy parameter $E_{cl}\to 2/g^2[1+(\frac{K_B T_{\max} L}{m\hbar c})^2] = 2m^2\pi^2/L^2$ and the configuration tends to the top of the potential $V(\varphi)$,

$$\varphi^t = \frac{\pi}{2}, \tag{12}$$

and for $k\to 1$, i.e., $E_{cl}\to 0$, the solution becomes

$$\varphi^0(x,\tau) = \arccos\left\{\tanh\left[\pm\frac{2(x-v\tau)}{g\sqrt{1+(v/c)^2}}\right]\right\}. \tag{13}$$

Considering $\sqrt{\frac{1-\cos(\varphi^0/2)}{1+\cos(\varphi^0/2)}}=\tan(\frac{\varphi^0}{2})=\exp[\mp\frac{2(x-v\tau)}{g\sqrt{1+(v/c)^2}}]$, we obtain the (anti)kinklike solutions

$$\varphi_{(\mathrm{anti})\mathrm{ins}}(x,\tau) = 2\arctan\left\{\exp\left[\mp\frac{(x-v\tau)}{g\sqrt{1+(v/c)^2}}\right]\right\}, \tag{14}$$

where the "+" sign means the instanton solution, otherwise, the so-called anti-instanton solution.

The tunneling amplitude of the domain wall is

$$\Gamma(T) \sim \exp\left(-\frac{W(T)}{\hbar}\right), \tag{15}$$

where the Euclidean action for the periodic instanton configuration [Eq. (6)] is

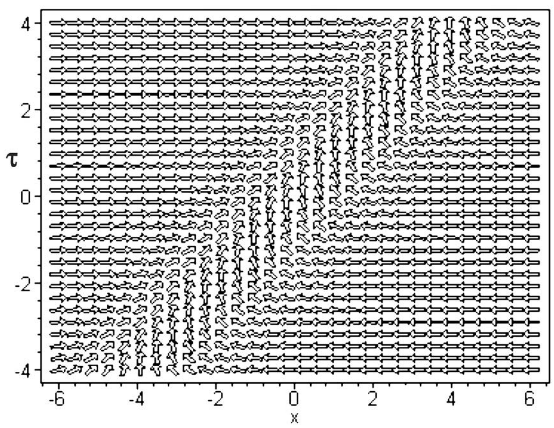

FIG. 1. Tunneling process of the domain wall for instanton solution (14). The variables along x and τ axes are in units of $g\sqrt{1+(v/c)^2}/2$ and $g\sqrt{1+(v/c)^2}/2v$, respectively.

$$W(T)=\int_0^p d\tau\int_0^L dx\pounds_e=\frac{8mJS\hbar^2}{gK_BT}[2E(k)-k'^2K(k)]\sqrt{1+\left(\frac{K_BTL}{m\hbar c}\right)^2}=\frac{8mJS\hbar^2}{gK_B}F(k)\sqrt{\frac{1}{T^2}+\left(\frac{K_BL}{m\hbar c}\right)^2}, \quad (16)$$

where $k'=\sqrt{1-k^2}$ and $E(k)$ denotes the complete elliptic integral of the second kind.

The function $F(k)=2E(k)-k'^2K(k)$ is monotonic and increases from $\pi/2$ to 2 for $0<k<1$. Thus it can be treated as a constant when analyzing the dependence relation of the Euclidean action W on T and L. For the maximal temperature permitted, we have the smallest action, consequently, the largest tunneling amplitude. The lower the temperature is, the smaller the tunneling amplitude of the domain wall is. In the limit of $T\to 0$, we have an infinite action, implying that the energy splitting of the degenerate ground state, which is proportional to Γ, is negligibly small, in accord with the existing result in literature[12] [in the limit of $\Delta V\to 0$ of Eq. (24) therein]. Similarly, the longer the spin chain is, the smaller the tunneling amplitude is. In the limit of $L\to\infty$, we also have an infinite action. But for a finite-length spin chain at finite temperature, it is possible to have a finite action and consequently, a finite-energy splitting of the degenerate orientations of the magnetization.

The evolution of the spin vectors as a function of spatial and imaginary time coordinates for the instanton configuration [Eq. (14)] is shown in Fig. 1, where the macroscopic quantum coherent tunneling of domain wall can be seen clearly. Equation (6) is a periodic instanton configuration which is responsible for the quantum tunneling at finite temperature.[3,4] The tunneling process is shown in Fig. 2. We want to note here that it is a virtual process which occurs only in the potential barrier with imaginary time; see also Fig. 3.2 of Ref. 6. In real time, the spin vectors go directly between $+X$ and $-X$ states.

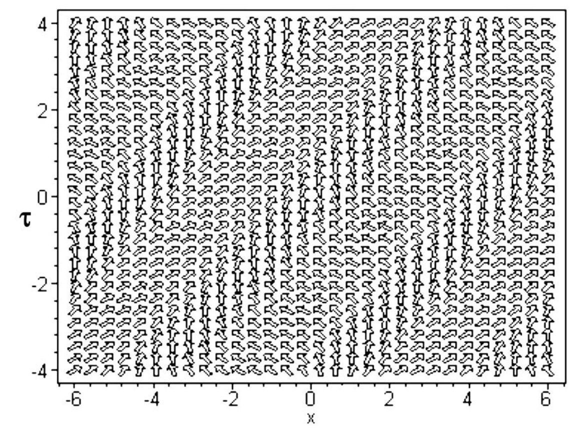

FIG. 2. Tunneling process of the domain wall for periodic instanton solution (6) with $k=0.9$. The units of the variables along x and τ axes are the same as those in Fig. 1.

IV. BREATHERLIKE PERIODIC INSTANTON

If we rescale the coordinates by $\tilde{x}=\frac{\sqrt{2}x}{g}$ and $\tilde{\tau}=\frac{\sqrt{2}c\tau}{g}$, the Euclidean field [Eq. (5)] reduces to

$$\frac{\partial^2\varphi}{\partial\tilde{x}^2}+\frac{\partial^2\varphi}{\partial\tilde{\tau}^2}=\sin(2\varphi). \quad (17)$$

Taking the ansatz that $\varphi=2\arctan[X(\tilde{x})/T(\tilde{\tau})]$,[18] we have

$$\left(\frac{\partial X}{\partial\tilde{x}}\right)^2=\alpha X^4+\beta X^2+\gamma,$$

$$\left(\frac{\partial T}{\partial\tilde{\tau}}\right)^2=\alpha T^4+(1-\beta)T^2+\gamma, \quad (18)$$

where α, β, and γ are integration constants.

Letting $\beta=1/l^2$ with $|l|<1$, we get the breatherlike configuration by direct integration of Eq. (18) in the case that $\alpha>0$ and $\gamma=0$

$$\varphi_b(x,\tau)=2\arctan\left\{\frac{1}{\sqrt{1-l^2}}\frac{\sin[\Omega_l\tau]}{\sinh\left[\frac{\sqrt{2}x}{g|l|}\right]}\right\}, \quad (19)$$

where the auxiliary parameter α is absent in the solution.

Breather is a localized solution with spatial width $\sqrt{2}g|l|<\sqrt{2}g$ and is considered as a bound pair configuration in soliton theory.[18] The instanton and the anti-instanton oscillate with respect to one another with the frequency

$$\Omega_l=\frac{c\sqrt{2(1-l^2)}}{g|l|}. \quad (20)$$

But, here, we consider it as the periodic instanton configuration with the period

$$\frac{\hbar}{K_BT}=\frac{2\pi}{\Omega_l}=\frac{\sqrt{2}\pi g|l|}{c\sqrt{(1-l^2)}}, \quad (21)$$

which determines entirely the parameter l and therefore the configuration at given temperature T.

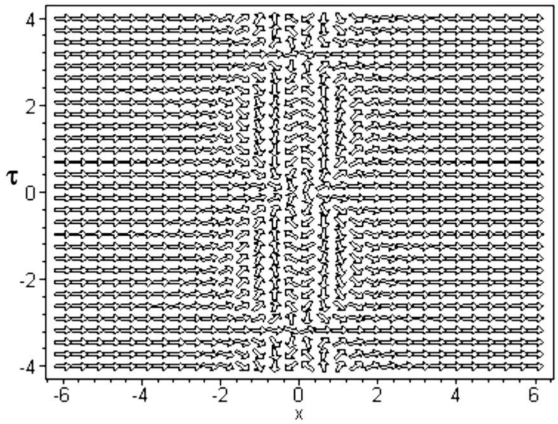

FIG. 3. Evolution of the spin vectors for breatherlike periodic instanton (19) with $l=\sqrt{0.5}$. The variables along x and τ axes are in units of $g/\sqrt{2}$ and $g/\sqrt{2}c$, respectively.

$\varphi_b(x,\tau)$ describes periodic instanton–anti-instanton pair configuration for an infinite-length spin chain. The corresponding time evolution of the giant spin vectors is shown in Fig. 3.

V. INSTANTON–ANTI-INSTANTON PAIR CONFIGURATIONS

Assuming $\beta=1/[1+(v/c)^2]$, we obtain the following configuration by direct integration of Eq. (18) in the case that $\alpha<0$ and $\gamma=0$:

$$\varphi_c(x,\tau) = 2\arctan\left\{\frac{c\cosh\left[\frac{\sqrt{2}v\tau}{g\sqrt{1+(v/c)^2}}\right]}{v\cosh\left[\frac{\sqrt{2}x}{g\sqrt{1+(v/c)^2}}\right]}\right\}, \tag{22}$$

where the auxiliary parameter α is also absent in the solution.

With the help of the mathematical formula $\arctan A+\arctan\frac{1}{A}=\arctan A+\operatorname{arccot} A=\pi/2(A>0)$,[19] we have

$$\varphi_c(x,\tau) = \pi - 2\arctan\left\{\frac{v\cosh\left[\frac{\sqrt{2}x}{g\sqrt{1+(v/c)^2}}\right]}{c\cosh\left[\frac{\sqrt{2}v\tau}{g\sqrt{1+(v/c)^2}}\right]}\right\}. \tag{23}$$

We then consider its asymptotic behavior in time following Ref. 20 in soliton theory. First, when time approaches $-\infty$, we have

$$\varphi_c(x,\tau) \to 2\arctan\left\{\frac{c\exp\left[-\frac{\sqrt{2}v\tau}{g\sqrt{1+(v/c)^2}}\right]}{2v\cosh\left[\frac{\sqrt{2}x}{g\sqrt{1+(v/c)^2}}\right]}\right\}. \tag{24}$$

For $x\to-\infty$,

$$\varphi_c(x,\tau) \to 2\arctan\left\{\frac{c}{v}\exp\left[\frac{\sqrt{2}(x-v\tau)}{g\sqrt{1+(v/c)^2}}\right]\right\} = 2\arctan\left\{\exp\left[\frac{\sqrt{2}\left[x-v\left(\tau+\frac{\Delta}{2}\right)\right]}{g\sqrt{1+(v/c)^2}}\right]\right\}, \tag{25}$$

where

$$\Delta = g\ln(v/c)\sqrt{2[1+(v/c)^2]}/v. \tag{26}$$

For $x\to+\infty$,

$$\varphi_c(x,\tau) \to 2\arctan\left\{\exp\left[-\frac{\sqrt{2}\left[x+v\left(\tau+\frac{\Delta}{2}\right)\right]}{g\sqrt{1+(v/c)^2}}\right]\right\}. \tag{27}$$

The solution therefore corresponds to an instanton–anti-instanton pair traveling with opposite velocities and approaching one another in the distant past.

In the same manner, we obtain its asymptotic behavior when time approaches $+\infty$,

$$\varphi_c(x,\tau) \to \pi - 2\arctan\left\{\frac{2v\cosh\left[\frac{\sqrt{2}x}{g\sqrt{1+(v/c)^2}}\right]}{c\exp\left[\frac{\sqrt{2}v\tau}{g\sqrt{1+(v/c)^2}}\right]}\right\}. \tag{28}$$

For $x\to-\infty$,

$$\varphi_c(x,\tau) \to \pi - 2\arctan\left\{\exp\left[-\frac{\sqrt{2}\left[x+v\left(\tau-\frac{\Delta}{2}\right)\right]}{g\sqrt{1+(v/c)^2}}\right]\right\}. \tag{29}$$

For $x\to+\infty$,

$$\varphi_c(x,\tau) \to \pi - 2\arctan\left\{\exp\left[\frac{\sqrt{2}\left[x-v\left(\tau-\frac{\Delta}{2}\right)\right]}{g\sqrt{1+(v/c)^2}}\right]\right\}. \tag{30}$$

In the distant future, the solution corresponds to the same instanton–anti-instanton pair with the same shapes and velocities but at opposite position. Another important change from the initial configuration is the time delay Δ which remains as the sole residual effect of the tunneling process. The corresponding time evolution of the giant spin vectors is shown in Fig. 4.

For $\gamma\neq0$, we obtain the corresponding periodic configuration

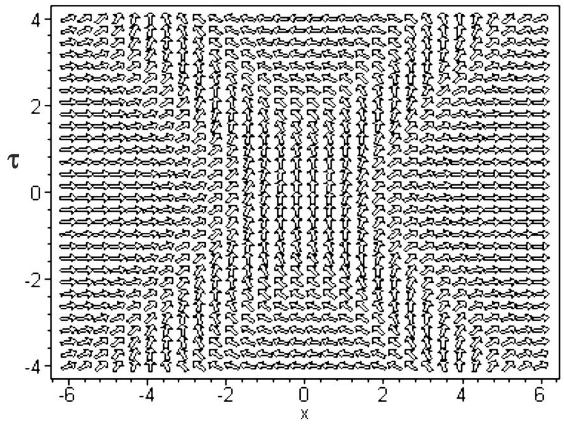

FIG. 4. Evolution of the spin vectors for instanton–anti-instanton pair configuration (22) with $v/c=0.5$. The variables along x and τ axes are in units of $g/\sqrt{2}$ and $g/\sqrt{2}v$, respectively.

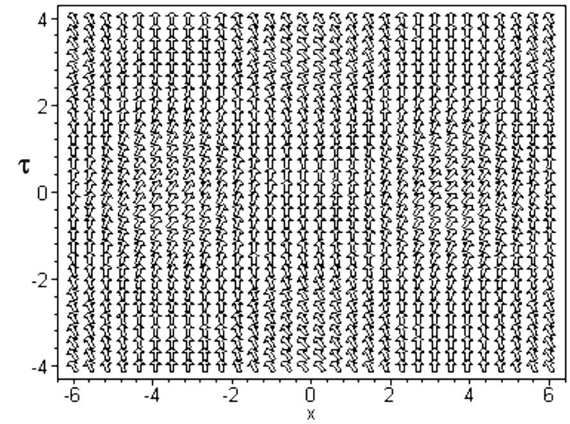

FIG. 5. Evolution of the spin vectors for periodic instanton–anti-instanton pair configuration (31) with $k_1=k_2=0.9$. The units of the variables along x and τ axes are the same with those in Fig. 4.

$$\varphi_p(x,\tau)=2\arctan\left\{\frac{c\sqrt{2-k_2^2}}{v\sqrt{2-k_1^2}}\frac{dn\left[\dfrac{\sqrt{2}x}{g\sqrt{[1+(v/c)^2](2-k_1^2)}},k_1\right]}{dn\left[\dfrac{\sqrt{2}v\tau}{g\sqrt{[1+(v/c)^2](2-k_2^2)}},k_2\right]}\right\}, \tag{31}$$

where $dn\ (x,k)=\sqrt{1-k^2\mathrm{sn}^2(x,k)}$ is also named Jacobian elliptic function.[16] For the above solution we have the restriction such that

$$\frac{k_1'}{(2-k_1^2)}=\frac{k_2'(v/c)}{(2-k_2^2)}. \tag{32}$$

In the limit $k_i\to 0$ $(i=1,2)$, the solution exists only for $v=c$ and therefore tends to the trivial configuration $\varphi^t=\frac{\pi}{2}$. In the limit $k_i\to 1$, the solution reduces to the configuration which is seen to be the same with solution (22) for any velocity v. Thus solution (31) is the periodic instanton–anti-instanton pair configuration for the quantum tunneling at finite temperature. The corresponding time evolution of the giant spin vectors is shown in Fig. 5.

The parameters k_1 and k_2 are to be determined from the periodic boundary condition

$$\frac{\sqrt{2}L}{g\sqrt{[1+(v/c)^2](2-k_1^2)}}=2m_1K(k_1), \tag{33}$$

where $m_1=1,2,3,\ldots$, and the period of the periodic instanton at given temperature T

$$\frac{\hbar}{K_BT}=\frac{2K(k_2)g\sqrt{[1+(v/c)^2](2-k_2^2)}}{\sqrt{2}v}. \tag{34}$$

Generally, if there are several solutions for the equation of the instanton, one should take into account only the one with the least action, which will dominate Eq. (15), unless all or some of the solutions happen to have comparable actions, in which case one should take all relevant contributions to Γ into account.[1] However, those corresponding to the configurations in Secs. IV and V are unable to be obtained and are lacking in the present paper. These need further study in the future.

VI. SUMMARY

We in the present paper provide various instanton configurations of $1+1$ dimensions in a ferromagnetic spin chain with biaxial anisotropy, and the coherent tunneling process of domain walls is illustrated explicitly. The magnitude of the tunneling amplitude corresponding to the periodic instanton configuration [Eq. (6)] is obtained. Result shows that it is possible to have a finite-energy splitting of the degenerate orientations of the magnetization along easy axis for a finite-length spin chain at finite temperature.

ACKNOWLEDGMENTS

This work was supported by the NSF of China under Grants No. 10647118, No. 10775091, No. 90406017, and No. 60525417, and by the NKBRSF of China under Grant No. 2006CB921400. G.-P.Z. also acknowledges the support from the Natural Science Basic Research Plan in Henan Province of China (Contract No. 2007140009), and the Youth Fund of Henan Normal University (Contract No. 2006006).

APPENDIX: CALCULATION OF INSTANTON SOLUTION (6)

We look for the trial solution $\varphi(\xi)$ with $\xi=x-v\tau$. Then we have

$$\left[1+\left(\frac{v}{c}\right)^2\right]\left(\frac{d^2\varphi}{d\xi^2}\right)=\frac{2}{g^2}\sin(2\varphi). \tag{A1}$$

Multiplying with the term $(\frac{d\varphi}{d\xi})$ and integrating over the variable ξ, the equation of motion is seen to be

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2-\frac{V(\varphi)}{[1+(v/c)^2]}=-E_{cl}, \tag{A2}$$

where the integration constant E_{cl} plays the role of dimensionless energy density. We want to get the instanton configurations with the energy parameter confined in the region $0<E_{cl}<\frac{2}{g^2[1+(v/c)^2]}$, namely, $E_{cl}=\frac{2(1-k^2)}{g^2[1+(v/c)^2]}$ with $0<k<1$. Then we have

$$\left(\frac{d\varphi}{d\xi}\right)^2=\frac{4}{g^2[1+(v/c)^2]}(k^2-\cos^2\varphi). \tag{A3}$$

Letting $\cos\varphi=k\cos\omega$, we get

$$\left(\frac{d\omega}{d\xi}\right)^2=\frac{4}{g^2[1+(v/c)^2]}(1-k^2\cos^2\omega). \tag{A4}$$

By direct integration[16] of the equation

$$\frac{d\omega}{\sqrt{1-k^2\cos^2\omega}}=\pm\frac{2d\xi}{g\sqrt{1+(v/c)^2}}, \tag{A5}$$

we have

$$\cos\omega=\mathrm{sn}\left[\pm\frac{2(x-v\tau-x_0)}{g\sqrt{1+(v/c)^2}},\ k\right], \tag{A6}$$

then the instanton solution [Eq. (6)] is obtained.

[1] S. Coleman, in *The Whys of Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1979).
[2] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[3] J.-Q. Liang and H. J. W. Müller-Kirsten, Phys. Rev. D **46**, 4685 (1992).
[4] J.-Q. Liang and H. J. W. Müller-Kirsten, Phys. Rev. D **51**, 718 (1995).
[5] *Quantum Tunneling of Magnetization—QTM '94*, edited by L. Gunther and B. Barbara (Kluwer, Dordrecht, 1995).
[6] E. M. Chudnovsky and J. Tejada, *Macroscopic Quantum Tunneling of the Magnetic Moment* (Cambridge University Press, Cambridge, 1998).
[7] P. C. E. Stamp, Phys. Rev. Lett. **66**, 2802 (1991).
[8] E. M. Chudnovsky, O. Iglesias, and P. C. E. Stamp, Phys. Rev. B **46**, 5392 (1992).
[9] G. Tatara and H. Fukuyama, Phys. Rev. Lett. **72**, 772 (1994).
[10] H. B. Braun, J. Kyriakidis, and D. Loss, Phys. Rev. B **56**, 8129 (1997).
[11] J. Shibata and S. Takagi, Phys. Rev. B **62**, 5719 (2000).
[12] I. V. Krive and O. B. Zaslavskii, J. Phys.: Condens. Matter **2**, 9457 (1990).
[13] J.-Q. Liang, H. J. W. Müller-Kirsten, Jian-Ge Zhou, F. Zimmerschied, and F.-C. Pu, Phys. Lett. B **393**, 368 (1997).
[14] H. J. Mikeska, J. Phys. C **11**, L29 (1978).
[15] A. O. Caldeira and K. Furuya, J. Phys. C **21**, 1227 (1988).
[16] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer, Berlin, 1954).
[17] R. P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972).
[18] G. L. Lamb, *Elements of Soliton Theory* (Addison-Wesley, New York, 1980).
[19] I. S. Gradsbteyn and L. M. Ryzbik, *Table of Integrals, Series, and Products* (Elsevier, Singapore, 2004).
[20] J. Rubinstein, J. Math. Phys. **11**, 258 (1970).