Bound of entanglement of assistance and monogamy constraints

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We investigate the entanglement of assistance which quantifies capabilities of producing pure bipartite entangled states from a pure tripartite state. The lower bound and upper bound of entanglement of assistance are obtained. In the light of the upper bound, monogamy constraints are proved for arbitrary n-qubit states.

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I. INTRODUCTION

In quantum-information theory, entanglement is a vital resource for some practical applications such as quantum cryptography, quantum teleportation, and quantum computation [1,2]. During the last decade, this inspired a great deal of effort for detecting and quantifying the entanglement [3–11]. On the other hand, the creation and distribution of entanglement are also of central interest in quantum-information processing. More especially, the distribution of bipartite entanglement is a key ingredient for performing certain quantum-information processing tasks such as teleportation.

One of the methods, generating bipartite entanglement, is the entanglement of assistance that is defined in Refs. [12,13]. It quantifies the entanglement which could be created by reducing a multipartite entangled state to an entangled state with fewer parties (e.g., bipartite) via measurements. Such producing of entanglement, also called "assisted entanglement," is a special case of the *localizable entanglement* [14], which is especially important for quantum communication, where quantum repeaters are needed to establish bipartite entanglement over a long length scale [15]. For a pure $2\otimes 2\otimes n$ state, the analytical formula of entanglement of assistance has been derived by Laustsen *et al.* [16], whereas the calculation of entanglement of assistance is not easy for a general pure tripartite state [17].

In this paper, we explore the entanglement of assistance for a general pure tripartite state in terms of I concurrence [18]. We obtain a lower bound of entanglement of assistance, which is also the lower bound of a tripartite entanglement measure, the entanglement of collaboration. This may help to characterize the localizable entanglement. Furthermore, an upper bound is also obtained. Deducing from the upper bound of entanglement of assistance, we find a proper form of entanglement monogamy inequality for arbitrary n-qubit states, which is analogous to the monogamy constraints for concurrence proposed by Coffman *et al.* [19] and proven by Osborne and Verstraete [20] for the general case.

The paper is organized as follows. In Sec. II, we derive a lower bound and upper bound of entanglement of assistance for pure tripartite states. In Sec. III, monogamy constraints are proved in terms of this upper bound. Finally in Sec. IV, we conclude with a discussion of our results.

II. BOUND OF ENTANGLEMENT OF ASSISTANCE

We consider a pure $(d_1\times d_2\times N)$ tripartite state shared by three parties referred to as Alice, Bob, and Charlie, who performs a measurement on his party to yield a known bipartite entangled state shared by Alice and Bob. Charlie's aim is to maximize the entanglement of the state between Alice and Bob. This maximum average entanglement that he can create is called entanglement of assistance, which was originally defined in terms of entropy of entanglement [12,13]. In this paper, we define entanglement of assistance in terms of the entanglement measure I concurrence $E_a(|\psi\rangle_{ABC})\equiv E_a(\rho_{AB})\equiv \max\Sigma_i p_i C(|\phi_i\rangle_{AB})$, which is maximized over all possible pure-state decompositions of $\rho_{AB}=\mathrm{Tr}_C[|\psi\rangle_{ABC}\langle\psi|]=\Sigma_i p_i|\phi_i\rangle_{AB}\langle\phi_i|$. By applying the method in Ref. [4], we can obtain the lower bound of entanglement of assistance for pure tripartite states.

For any given pure-state decomposition of ρ_{AB}, $\rho_{AB}=\Sigma_i p_i|\phi_i\rangle\langle\phi_i|$, we have

$$\begin{aligned}E_a(|\psi\rangle_{ABC}) &= \max\sum_i p_i C(|\phi_i\rangle_{AB})\\ &= \max\sum_i p_i\sqrt{\sum_{mn}|\langle\phi_i^*|S_{mn}|\phi_i\rangle|^2}\\ &\geq \max\sqrt{\sum_{mn}\left(\sum_i p_i|\langle\phi_i^*|S_{mn}|\phi_i\rangle|\right)^2},\end{aligned} \tag{1}$$

where $S_{mn}=L_m\otimes L_n$, $L_m, m=1,\dots,d_1(d_1-1)/2$, and $L_n, n=1,\dots,d_2(d_2-1)/2$ are the generators of groups SO(d_1) and SO(d_2), respectively. The inequality holds according to the Minkowski inequality $[\Sigma_{i=1}(\Sigma_k x_i^k)^p]^{1/p}\leq\Sigma_k[\Sigma_{i=1}(x_i^k)^p]^{1/p}, p>1$.

For any given decomposition $\rho_{AB}=\Sigma_i p_i|\phi_i\rangle\langle\phi_i|$, we can take the matrix notation [21] and rewrite the state as $\rho_{AB}=\Phi P\Phi^\dagger$, where P is a diagonal matrix with $P_{ii}=p_i$ and Φ is a unitary matrix whose columns are the vectors ϕ_i. Consider the eigenvalue decomposition $\rho_{AB}=\Psi M\Psi^\dagger$, where M is a diagonal matrix whose diagonal elements are the eigenvalues of ρ_{AB} and the columns of the matrix Ψ correspond to the eigenvectors of ρ_{AB}. Moreover, we can get the relation $\Phi P^{1/2}=\Psi M^{1/2}U$, where U is a right-unitary matrix. Thus we can rewrite inequality (1) as

$$\begin{aligned}E_a(\rho_{AB}) &\geq \max\sqrt{\sum_{mn}\left(\sum_i|\Phi^T P^{1/2}S_{mn}P^{1/2}\Phi|_{ii}\right)^2}\\ &= \max\sqrt{\sum_{mn}\left(\sum_i|U^T M^{1/2}\Psi^T S_{mn}\Psi M^{1/2}U|_{ii}\right)^2}\\ &= \max\sqrt{\sum_{mn}\left(\sum_i|U^T A_{mn}U|_{ii}\right)^2},\end{aligned} \tag{2}$$

where $A_{mn}=M^{1/2}\Psi^T S_{mn}\Psi M^{1/2}$.

In terms of the Cauchy-Schwarz inequality $(\Sigma_i x_i^2)^{1/2}(\Sigma_i y_i^2)^{1/2} \geq \Sigma_i x_i y_i$, the inequality

$$E_a(\rho_{AB}) \geq \max \sqrt{\sum_{mn}\left(\sum_i |U^T A_{mn} U|_{ii}\right)^2}\sqrt{\sum_{mn}|z_{mn}|^2}$$
$$\geq \max \sum_i \left|U^T\left(\sum_{mn} z_{mn}A_{mn}\right)U\right|_{ii} \quad (3)$$

is implied for any $z_{mn}=y_{mn}\exp(i\theta_{mn})$, with $y_{mn}\geq 0$ and $\Sigma_{mn}y_{mn}^2=1$. Since $\Sigma_{mn}z_{mn}A^{mn}$ is a symmetric matrix, we can always find a unitary matrix U such that $\Sigma_i|U^T(\Sigma_{mn}z_{mn}A_{mn})U|_{ii}=\|\Sigma_{mn}z_{mn}A^{mn}\|$ as shown in Ref. [22], where $\|\cdot\|$ stands for the trace norm defined by $\|G\| = \mathrm{Tr}(GG^\dagger)^{1/2}$.

For an arbitrary unitary matrix V, we have

$$\begin{aligned}
&\sum_i \left|V^T\left(\sum_{mn} z_{mn}A_{mn}\right)V\right|_{ii} \\
&= \sum_i \left|V^T(U^{-1})^T U^T\left(\sum_{mn} z_{mn}A_{mn}\right)UU^{-1}V\right|_{ii} \\
&= \sum_i |V^T(U^{-1})^T \mathrm{Diag}(\lambda_1,\lambda_2\cdots)U^{-1}V|_{ii} \leq \sum_{ij}|(U^{-1}V)_{ij}|^2\lambda_i \\
&= \sum_i \lambda_i,
\end{aligned}$$

where $\lambda_i(z)$'s, dependent on the choice of the y and θ, are the singular values of the matrix $\mathcal{T}=\Sigma_{mn}z_{mn}A^{mn}$, i.e., the square roots of the eigenvalues of the positive Hermitian matrix $\mathcal{T}\mathcal{T}^\dagger$. Therefore, the maximum of Eq. (3) is given by $\max_{z\in\mathbf{C}}[\Sigma_i\lambda_i(z)]=\max_{z\in\mathbf{C}}\|\Sigma_{mn}z_{mn}A^{mn}\|$. Hence, we arrive at the lower bound of entanglement of assistance for a pure tripartite state as following:

$$E_a(\rho_{AB}) \geq \max_{z\in\mathbf{C}}\left\|\sum_{mn} z_{mn}A_{mn}\right\|. \quad (4)$$

Moreover, it has been shown by Gour and Spekkens [23] that, for tripartite states, entanglement of collaboration [24,25] is greater than or equal to entanglement of assistance in terms of a given entanglement measure. Therefore, our lower bound is also the one for entanglement of collaboration. Our bound may help to characterize localizable entanglement. For a pure $2\times 2\times N$ state, this lower bound is consistent with the result of Ref. [16].

Generally, for each set $\{y_{mn},\theta_{mn}\}$, we can obtain a lower bound of $E_a(\rho_{AB})$, which can be tightened by numerical optimization. In fact, there is always one matrix A_{mn} that gives the main contribution to the right-hand side of Eq. (4). Hence, the singular values of this matrix provide a simple algebraic approximation for the lower bound of $E_a(\rho_{AB})$. As an example, we consider a state $|\phi\rangle_{ABC}=\sqrt{a}|000\rangle+\sqrt{b}|111\rangle+\sqrt{c}|222\rangle$ with $a+b+c=1$. Without loss of generality, we assume that $a\geq b\geq c\geq 0$. Then, we get a purely algebraic estimate for the lower bound of $E_a(\rho_{AB})$, $2\sqrt{ab}$, without numerical optimization. In the case $a=b=c=1/3$, a lower bound of $E_a(\rho_{AB})$, $4\sqrt{3}/9$, is obtained through an optimization.

We can also obtain the upper bound of entanglement of assistance. From the definition of entanglement of assistance, we have

$$\begin{aligned}
[E_a(\rho_{AB})]^2 &= \left[\max\sum_i p_i C(|\phi_i\rangle_{AB})\right]^2 \\
&\leq \max\sum_i [\sqrt{p_i}C(|\phi_i\rangle_{AB})]^2\sum_i(\sqrt{p_i})^2 \\
&= \max\sum_i 2p_i[1-\mathrm{Tr}(\rho_i^A)^2] \leq 2(1-\mathrm{Tr}\,\rho_A^2),
\end{aligned}$$

where $\rho_i^A=\mathrm{Tr}_B|\phi_i\rangle_{AB}\langle\phi_i|$. The first inequality holds according to the Cauchy-Schwarz inequality [26]; the last one, which has also been proved in Ref. [27], holds due to the convex property of Tr ρ_A^2. Define the upper bound as the tangle of assistance $\tau_a(\rho_{AB})\equiv\max\Sigma_i p_i[C(|\phi_i\rangle_{AB})]^2$.

In addition to the tangle of assistance being the upper bound of entanglement of assistance, the tangle of assistance for n-qubit states also exhibits some proper forms of monogamy inequality. In fact, not only does the concurrence satisfy the monogamy constraints for n-qubit state [19,20], but also the entanglement of assistance exhibit monogamy constraints for n-qubit pure states [28–30]. We will show below the monogamy inequalities in the light of the tangle of assistance [31].

III. MONOGAMY INEQUALITY

Consider a pure tripartite state $|\Psi\rangle_{ABC}$. The tangle of assistance is defined by $\tau_a(|\Psi\rangle_{ABC})=\max_{\{p_x,|\psi_x\rangle\}}\Sigma_x p_x[C(|\psi_x\rangle)]^2 =\max_{\{p_x,|\psi_x\rangle\}}\Sigma_x p_x S_2[\mathrm{Tr}_B(|\psi_x\rangle\langle\psi_x|)]$, where the linear entropy $S_2[\rho]=2[1-\mathrm{Tr}(\rho)^2]$ and the maximum runs over all pure-state decompositions $\{p_x,|\psi_x\rangle\}$ of $\rho_{AB}=\mathrm{Tr}_C(|\Psi\rangle_{ABC}\langle\Psi|) =\Sigma_x p_x|\psi_x\rangle\langle\psi_x|$. In the case of pure state ρ_{AB}, the tangle of assistance is the square of concurrence of this state.

Theorem 1. For an arbitrary n-qubit state, the tangle of assistance satisfies

$$\tau_a(\rho_{A_1A_2})+\tau_a(\rho_{A_1A_3})+\cdots+\tau_a(\rho_{A_1A_n})\geq\tau_a(\rho_{A_1(A_2A_3\cdots A_n)}), \quad (5)$$

where $\tau_a(\rho_{A_1(A_2A_3\cdots A_n)})$ denotes the tangle of assistance in the bipartite partition $A_1|A_2A_3\cdots A_n$.

Proof. First of all, we prove the following inequality:

$$\tau_a(\rho_{AB})+\tau_a(\rho_{AC})\geq\tau_a(\rho_{A(BC)}), \quad (6)$$

for arbitrary tripartite states ρ_{ABC} in $2\times 2\times 2^{n-2}$ system.

We first prove Eq. (6) for pure states. In this case, due to the local-unitary invariance of $\tau_a(\rho_{AC})$, we can rotate the basis of subsystem C into the local Schmidt basis $|V_k\rangle$, $k=1,\ldots,4$, given by the eigenvectors of $\rho_C=Tr_{AB}(\rho_{ABC})$. In this way, we can regard the 2^{n-2}-dimensional qudit C as an effective four-dimensional qudit. Therefore, we simply need to prove Eq. (6) for a $2\times 2\times 4$ pure state ABC.

For pure states of a tripartite system ABC of two qubits A and B and a four-level system C, we have $\tau_a(\rho_{A(BC)}) -\tau_a(\rho_{AC})=S_2(\rho_A)-\max_{\{p_j,|\phi_j\rangle\}}\Sigma_j p_j S_2[\mathrm{Tr}_C(|\phi_j\rangle\langle\phi_j|)]$, where $\Sigma_j p_j|\phi_j\rangle\langle\phi_j|=\rho_{AC}$. It can be shown that any pure-state decomposition of ρ_{AC} can be realized by positive-operator-valued measures (POVMs) $\{M_x\}$ performed by Bob, the rank of which is 1 (for more details see [17,32]). Therefore, we get the following expression:

$$\tau_a(\rho_{AC}) = \max_{\{M_x\}} \sum_x p_x S_2(\rho_x), \tag{7}$$

where the maximum runs over all rank-1 POVMs on Bob's system, $p_x = \mathrm{Tr}(I_A \otimes M_x \rho_{AB})$ is the probability of outcome x, and $\rho_x = \mathrm{Tr}_B(I_A \otimes M_x \rho_{AB})/p_x$ is the posterior state in Alice's subsystem. For convenience, we take the definition $I(\rho_{AB}) := S_2(\rho_A) - \max_{\{M_x\}} \sum_x p_x S_2(\rho_x)$. By comparing $I(\rho_{AB})$ to Eq. (6) for pure tripartite states, we see that it is sufficient to prove the inequality $I(\rho_{AB}) \le \tau_a(\rho_{AB})$ for all two-qubit states ρ_{AB}.

We first derive a computable formula for $I(\rho_{AB})$. Any bipartite quantum state ρ_{AB} may be written as

$$\rho_{AB} = \Lambda \otimes I_B(|V_{B'B}\rangle\langle V_{B'B}|), \tag{8}$$

where $V_{B'B}$ is the symmetric two-qubit purification of the reduced density operator ρ_B on an auxiliary qubit system B' and Λ is a qubit channel from B' to A. Deducing from Eq. (7), we have $\rho_x = \mathrm{Tr}_B(I_A \otimes M_x \rho_{AB})/p_x = \mathrm{Tr}_B[(I_A \otimes M_x) \times (\Lambda \otimes I_B)|V_{B'B}\rangle\langle V_{B'B}|]/p_x = \Lambda[\mathrm{Tr}_B(I_A \otimes M_x |V_{B'B}\rangle\langle V_{B'B}|)]/p_x$. Since the rank of M_x is 1, $\mathrm{Tr}_B(I_A \otimes M_x |V_{B'B}\rangle\langle V_{B'B}|)]$ is a pure state. Moreover, all pure-state decompositions of $\rho_B' = \mathrm{Tr}_B(|V_{B'B}\rangle\langle V_{B'B}|) = \rho_B$ can be realized by the rank-1 POVM measurements $\{M_x\}$ operating on subsystem B of $|V_{B'B}\rangle\langle V_{B'B}|$. Hence, $I(\rho_{AB})$ satisfies

$$I(\rho_{AB}) = S_2[\Lambda(\rho_B)] - \max_{\{p_x, |\psi_x\rangle\}} \sum_x p_x S_2[\Lambda(|\psi_x\rangle)], \tag{9}$$

where the maximum runs over all pure-state decompositions $\{p_x, |\psi_x\rangle\}$ of ρ_B such that $\sum_x p_x |\psi_x\rangle\langle\psi_x| = \rho_B$.

The action of a qubit channel Λ on a single-qubit state $\rho = (I + \mathbf{r}\cdot\boldsymbol{\sigma})/2$, where $\boldsymbol{\sigma}$ is the vector of Pauli operators, may be written as $\Lambda(\rho) = [I + (\mathbf{L}\mathbf{r} + \mathbf{l})\cdot\boldsymbol{\sigma}]/2$, where $\mathbf{L}$ is a 3×3 real matrix and $\mathbf{l}$ is a three-dimensional vector. In this Pauli basis, the possible pure-state decompositions of ρ_B are represented by all possible sets of probabilities $\{p_j\}$ and unit vectors $\{\mathbf{r}_j\}$ such that $\sum_j p_j \mathbf{r}_j = \mathbf{r}_B$, where $(I + \mathbf{r}_B\cdot\boldsymbol{\sigma})/2 = \rho_B$. In terms of the Block representation of one-qubit states, the linear entropy S_2 is given by $S_2[(I + \mathbf{r}\cdot\boldsymbol{\sigma})/2] = 1 - |\mathbf{r}|^2$. In this way, we get the following equation: $S_2[\Lambda(I + \mathbf{r}\cdot\boldsymbol{\sigma})/2] = 1 - (\mathbf{L}\mathbf{r} + \mathbf{l})^T(\mathbf{L}\mathbf{r} + \mathbf{l})$.

Substituting $\mathbf{r}_j = \mathbf{r}_B + \mathbf{x}_j$, one can easily check that Eq. (9) reduces to the following one whose value is determined by $\{p_j, \mathbf{x}_j\}$ subject to the conditions $\sum_j p_j \mathbf{x}_j = 0$ and $|\mathbf{r}_B + \mathbf{x}_j| = 1$:

$$\begin{aligned} I(\rho_{AB}) &= S_2[\Lambda(\rho_B)] - \max_{\{p_j, \mathbf{x}_j\}} \sum_j p_j S_2\left[\Lambda\left(\frac{I + (\mathbf{r}_B + \mathbf{x}_j)\cdot\boldsymbol{\sigma}}{2}\right)\right] = 1 \\ &\quad - (\mathbf{L}\mathbf{r}_B + \mathbf{l})^T(\mathbf{L}\mathbf{r}_B + \mathbf{l}) - \max_{\{p_j, \mathbf{x}_j\}} \sum_j p_j \\ &\quad \times \{1 - [\mathbf{L}(\mathbf{r}_B + \mathbf{x}_j) + \mathbf{l}]^T[\mathbf{L}(\mathbf{r}_B + \mathbf{x}_j) + \mathbf{l}]\} \\ &= \min_{\{p_j, \mathbf{x}_j\}} \sum_j p_j (\mathbf{x}_j^T \mathbf{L}^T \mathbf{L} \mathbf{x}_j). \end{aligned} \tag{10}$$

Without loss of generality, we assume that $\mathbf{L}^T\mathbf{L}$ is diagonal with diagonal elements $\lambda_x \le \lambda_y \le \lambda_z$. The constrains $|\mathbf{r}_B + \mathbf{x}_j| = 1$ lead to the identities $(\mathbf{x}_j^x)^2 = 1 - |\mathbf{r}_B|^2 - 2\mathbf{r}_B^T \mathbf{x}_j - (\mathbf{x}_j^y)^2 - (\mathbf{x}_j^z)^2$. Substituting this into Eq. (10), we get $I(\rho_{AB}) = \lambda_x(1 - |\mathbf{r}_B|^2) + \min_{\{p_j, \mathbf{x}_j\}} \sum_j p_j[(\lambda_y - \lambda_x)(\mathbf{x}_j^y)^2 + (\lambda_z - \lambda_x)(\mathbf{x}_j^z)^2]$. This expression is obviously minimized by choosing $\mathbf{x}_j^z = \mathbf{x}_j^y = 0$ for all j. Then, from the condition $|\mathbf{r}_B + \mathbf{x}_j| = 1$, $\mathbf{x}_j^x$ have two solutions. The ensemble of two states corresponding to such two solutions can reach the minimum $\lambda_x(1 - |\mathbf{r}_B|^2)$.

As $S_2(\rho_B) = (1 - |\mathbf{r}_B|^2)$, we obtain the following computable expression: $I(\rho_{AB}) = \lambda_{\min} S_2(\rho_B)$. Note that a local filtering operation of the form $\rho_{AB}' = (I \otimes B)\rho_{AB}(I \otimes B^\dagger)/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]$ leaves $\mathbf{L}$ invariant and transforms $S_2(\rho_{B'}) = \det(B)^2/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]^2 S_2(\rho_B)$ [33].

If the local filtering operator B is invertible, we can get the conclusion that there does not exist a pure-state decomposition $\{q_j, |\psi_j\rangle\}$ of ρ_{AB}' such that $\tau_a(\rho_{AB}') > \det(B)^2/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]\tau_a(\rho_{AB})$ by the contradiction. For the case that the operator B is not invertible, such pure-state decomposition also does not exist. Furthermore, there exists exactly an optimal pure-state decomposition $\{p_i, |\phi_i\rangle\}$ of the state ρ_{AB} for $\tau_a(\rho_{AB})$ such that $\sum_i p_i C[(I \otimes B)(|\phi_i\rangle\langle\phi_i| I \otimes B^\dagger)/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]]^2 = \det(B)^2/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]^2 \tau_a(\rho_{AB})$. Therefore, the tangle of assistance $\tau_a(\rho_{AB}') = \det(B)^2/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]^2 \tau_a(\rho_{AB})$. Since $I(\rho_{AB}') = \det(B)^2/\mathrm{Tr}[(I \otimes B^\dagger B)\rho_{AB}]^2 \lambda_{\min} S_2(\rho_B)$, it transforms exactly in the same way as the tangle of assistance $\tau_a(\rho_{AB}')$ does. As there always exists a filtering operation for which $\rho_B' \propto I$, we can assume, without loss of generality, that $S_2(\rho_B) = 1$.

So let us consider ρ_{AB} with $\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \frac{1}{2}I$. In terms of Pauli operators, we can rewrite the pure state as follows: $(I \otimes B)|V_{B'B}\rangle\langle V_{B'B}|(I \otimes B^\dagger)/\mathrm{Tr}[(I \otimes B^\dagger B)|V_{B'B}\rangle\langle V_{B'B}|] = 1/4[I + \sum_i m_i I \otimes \sigma_i + \sum_i n_i \sigma_i \otimes I + \sum_{ij} O_{ij} \sigma_i \otimes \sigma_j]$, where σ_1, σ_2, and σ_3 are σ_x, σ_y, and σ_z, respectively. Then we get the conclusion, from its purity and unity reduced density, that $m_i = n_i = 0$ for all i and the 3×3 real matrix O is orthogonal. Thus we have $\rho_{AB} = \frac{1}{4}\Lambda \otimes I_B[I + \sum_{ij} O_{ij}\sigma_i \otimes \sigma_j] = \frac{1}{4}[I + \sum_i l_i \sigma_i \otimes I + \sum_{ij}(LO)_{ij}\sigma_i \otimes \sigma_j]$. As unitary operator U_1 satisfies the equation $U_1 \sigma_i U_1^\dagger = \sum_j P_{ij}\sigma_j$, where P is a real orthogonal 3×3 matrix, we can always find local-unitary operators, in terms of the theorem of singular value decomposition, so that $U_1 \otimes U_2 \rho_{AB} U_1^\dagger \otimes U_2^\dagger = \frac{1}{4}[I + \sum_i (lP)_i \sigma_i \otimes I + \sum_{ij}(QLOP)_{ij}\sigma_i \otimes \sigma_j] = \frac{1}{4}[I + \sum_i l_i' \sigma_i \otimes I + \sum_i (L')_{ii}\sigma_i \otimes \sigma_i]$, where Q and P are real orthogonal matrices and L' is a diagonal matrix with its diagonal elements the singular values of L. Because of the local-unitary invariance of $\tau_a(\rho_{AB})$ and $I(\rho_{AB})$, without loss of generality, we assume that $\rho_{AB} = \frac{1}{4}[I + \sum_i t_i \sigma_i \otimes I + \sum_i (R)_{ii}\sigma_i \otimes \sigma_i]$, where R is a diagonal matrix with its diagonal elements the singular values of L. Due to the positivity of

$$\rho_{AB} = \frac{1}{4}\begin{pmatrix} 1 + R_3 + t_3 & 0 & t_1 - it_2 & R_1 - R_2 \\ 0 & 1 - R_3 + t_3 & R_1 + R_2 & t_1 - it_2 \\ t_1 + it_2 & R_1 + R_2 & 1 - R_3 - t_3 & 0 \\ R_1 - R_2 & t_1 + it_2 & 0 & 1 + R_3 - t_3 \end{pmatrix},$$

the inequality $1 - t_1^2 - t_2^2 - t_3^2 \ge R_3^2$ must hold. Therefore we obtain

$$\tau_a(\rho_{AB}) \geq [C_a(\rho_{AB})]^2 \geq \mathrm{Tr}[\sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y \rho_{AB}]$$
$$= \frac{1}{16}[4 + 4(R_1^2 + R_2^2 + R_3^2) - 4(t_1^2 + t_2^2 + t_3^2)]$$
$$\geq \frac{1}{4}[R_1^2 + R_2^2 + 2R_3^2] \geq \lambda_{\min}(\mathbf{L}^T\mathbf{L}).$$

These inequalities imply that $I(\rho_{AB}) \leq \tau_a(\rho_{AB})$ for all two-qubit states ρ_{AB}, which then proves Eq. (6) for pure states.

Now we extend Eq. (6) to mixed state case. Consider the maximizing pure-state decomposition $\{p_x, |\psi_x\rangle\}$ for $\tau_a(\rho_{A(BC)})$. By applying the inequality Eq. (6) and taking into account the concavity of τ_a, we have $\tau_a(\rho_{A(BC)}) = \Sigma_x p_x \tau_a(\rho^x_{A(BC)}) \leq \Sigma_x p_x[\tau_a(\rho^x_{AB}) + \tau_a(\rho^x_{AC})] \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC})$, where $\rho^x_{A(BC)} = |\psi_x\rangle\langle\psi_x|$.

Let $C = C_1C_2$ be a $2 \times 2^{n-3}$ system and apply Eq. (6), then we get $\tau_a(\rho_{A(BC)}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC}) \leq \tau_a(\rho_{AB}) + \tau_a(\rho_{AC_1}) + \tau_a(\rho_{AC_2})$. Successively applying Eq. (6) to partitions of C, we obtain the inequality Eq. (5) by induction. ■

In fact, Eq. (5) turns out to be an equality for product states under partition $A|BC_1C_n$. For the generalized n-qubit GHZ states $|\psi\rangle = (|0\cdots0\rangle + |1\cdots1\rangle)/\sqrt{2}$, Eq. (5) is a strictly inequality.

IV. DISCUSSION

In summary, as an important quantity in quantum computation, the entanglement of assistance has been investigated in terms of I concurrence for pure tripartite states. We have obtained a lower bound of entanglement of assistance, which is also the lower bound of the tripartite entanglement measure, the entanglement of collaboration. Instead of great difficulty involved in computing the entanglement of collaboration, the lower bound Eq. (4) can be calculated in a numerical optimization to make a good estimate of entanglement of collaboration. Furthermore, this bound can also be reduced to a simple algebraic bound to provide a good approximation of entanglement of assistance without a numerical optimization. In addition to the lower bound, an upper bound is also obtained. In the light of the upper bound of entanglement of assistance, we find a proper form of entanglement monogamy inequality for arbitrary n-qubit states. We hope that this concise form of monogamy inequality may help to apprehend the nature of distributed entanglement.

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