Time evolution of the relative phase in two-component Bose-Einstein condensates with a coupling drive

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The dynamics of two-component Bose-Einstein condensates with coupling drive is studied using a pair of bosonic operators. We demonstrate that the effect of the coupling drive on the evolution of the relative phase of the two-component Bose-Einstein condensates depends on the initial relative phase difference between the two condensates. In some special initial phase differences ($\varphi = 0$ and π) the condensate density are not sinusoidal functions of time, and the interference pattern is not steady either. The coupling drive also reduces the amplitude and the period of the oscillation. Our theoretical result confirms the experimental observations reported in Phys. Rev. Lett. **81**, 1543 (1998).

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The experimental realization of Bose-Einstein condensates (BEC's) [1–5] of dilute gases has opened the possibility to test fundamental concepts in quantum mechanics, such as wavelike behavior and quantum phase interference, most notably analogous to the Josephson effects in superconductors and superfluid ^{3}He at ultralow temperatures. Recently, considerable theoretical attention has been applied to measure and monitor the time evolution of the phase of atomic BEC's. Though it is impossible in principle to set up a ''standard of phase'' in BEC's [6], the independent condensates are expected to possess a relative phase, the existence of which has been confirmed by the experimental observation of a spatially uniform interference pattern formed by condensates released from two independent traps [7]. Several papers were devoted to schemes [8–11] to measure the relative phase which is expected to give rise to a variety of interesting behaviors.

Recently, the relative phase and its subsequent time evolution in two BEC's were measured using a time-domain, separated-oscillatory-field condensate interferometer [5]. The two trapped condensates, created with a particular relative phase, are coupled by the coupling drive represented in the rotating-wave approximation. It has been found that the condensates retain a clear memory of their initial relative phase because of stable interference patterns, and the relative phase seems to possess a robustness which preserves coherence in the face of the ''phase-diffusing'' couplings to the environment. However, it is important to know how the coupling drive affects the evolution of the relative phase between two trapped condensates.

We consider a zero-temperature two-species Bose condensate system with weak nonlinear interatomic interactions and coupling drive. The coupling drive can be produced in experiment by a short two-photon pulse that transfers the atoms from one spin state to the other [12]. In the formalism of the second quantization, the Hamiltonian of such a system can be written as

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{int} + \hat{H}_{driv}, \tag{1}$$

$$\hat{H}_i = \int d\mathbf{x} \mathbf{\Psi}_i^\dagger(\mathbf{x}) \bigg[-\frac{\hbar^2}{2m}\nabla^2 + V_i(\mathbf{x}) + U_i(\mathbf{x})\mathbf{\Psi}_i^\dagger(\mathbf{x})\mathbf{\Psi}_i(\mathbf{x}) \bigg] \mathbf{\Psi}_i(\mathbf{x}), \tag{2}$$

$$\hat{H}_{int} = U_{12}\int d\mathbf{x} \mathbf{\Psi}_1^\dagger(\mathbf{x})\mathbf{\Psi}_2^\dagger(\mathbf{x})\mathbf{\Psi}_1(\mathbf{x})\mathbf{\Psi}_2(\mathbf{x}), \tag{3}$$

$$\hat{H}_{driv} = \frac{\hbar\Omega}{2}\int d\mathbf{x}[\mathbf{\Psi}_1^\dagger(\mathbf{x})\mathbf{\Psi}_2(\mathbf{x})e^{i\omega_{rf}t} + \mathbf{\Psi}_1(\mathbf{x})\mathbf{\Psi}_2^\dagger(\mathbf{x})e^{-i\omega_{rf}t}], \tag{4}$$

where $i=1$ and 2, and $\mathbf{\Psi}_i^\dagger(\mathbf{x})$ and $\mathbf{\Psi}_i(\mathbf{x})$ are the atomic field operators which create and annihilate atoms at position $\mathbf{x}$, respectively, and satisfy the commutation relation $[\mathbf{\Psi}_i(\mathbf{x}), \mathbf{\Psi}_j^\dagger(\mathbf{x}')] = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')$. $\hat{H}_1$ and $\hat{H}_2$ describe the evolution of each species in the absence of interspecies interactions, and $U_i(\mathbf{x}) = 4\pi\hbar^2 a_i/m$ (a_i and m are the intraspecies scattering lengths and the mass of the atom respectively). $\hat{H}_{int}$ describes interspecies collisions and $U_{ij} = 4\pi\hbar^2 a_{ij}/m$ ($a_{ij} = a_{ji}$ is the interspecies scattering lengths). $\hat{H}_{driv}$ here represents the coupling drive, which is in the rotating-wave approximation, and is characterized by the sum of the microwave and rf frequencies ω_{rf}, and by an effective *Rabi* frequency Ω [5].

We can obtain an effective two-mode boson Hamiltonian by means of [13,15,16]

$$H = \sum_{i=1,2}(\omega_0 \hat{a}_i^\dagger \hat{a}_i + q\hat{a}_i^{\dagger 2}\hat{a}_i^2) + 2\chi\hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_2 + g\Omega(\hat{a}_1^\dagger\hat{a}_2 e^{i\omega_{rf}t} + \hat{a}_2^\dagger\hat{a}_1 e^{-i\omega_{rf}t}), \tag{5}$$

where $\hat{a}_i^\dagger = \int d\mathbf{x}' \Phi_{iN}(\mathbf{x}')\mathbf{\Psi}_j^\dagger(\mathbf{x}')$ is a creative operator which creates a particle with distribution $\Phi_{iN}(\mathbf{x})$, and satisfies $[\hat{a}_i, \hat{a}_i^\dagger] = 1$ and the frequency

$$\omega_0=\int d\mathbf{x}\left[-\frac{\hbar^2}{2m}|\nabla\Phi_i(\mathbf{x})|^2+V_i(\mathbf{x})|\Phi_i(\mathbf{x})|^2\right], \qquad (6)$$

and the coupling constants q, g, and χ are defined by

$$q=U_0\int d\mathbf{x}[|\Phi_1(\mathbf{x})|^4+|\Phi_2(\mathbf{x})|^4], \qquad (7)$$

$$g=\int d\mathbf{x}[\Phi_1^\dagger(\mathbf{x})\Phi_2(\mathbf{x})+\Phi_1(\mathbf{x})\Phi_2^\dagger(\mathbf{x})], \qquad (8)$$

$$\chi=\frac{U_{12}}{2}\int d\mathbf{x}|\Phi_1(\mathbf{x})|^2|\Phi_2(\mathbf{x})|^2, \qquad (9)$$

which characterize the strength of the interatomic interaction in each condensate, the coupling drive, and the interspecies interaction, respectively. Here $\Phi_i(\mathbf{x})(i=1,2)$ denotes the stationary solution of the GP equations.

In order to obtain analytical eigenstates of the Hamiltonian [Eq. (5)], we introduce a new pair of bosonic operators $\hat{A}_1$ and $\hat{A}_2$ by the expressions

$$\hat{a}_1=\frac{1}{\sqrt{2}}(\hat{A}_1e^{igt}-i\hat{A}_2e^{-igt})e^{i(\omega_{rf}/2)t},$$

$$\hat{a}_2=\frac{1}{\sqrt{2}}(\hat{A}_1e^{igt}+i\hat{A}_2e^{-igt})e^{-i(\omega_{rf}/2)t}, \qquad (10)$$

where $\hat{A}_1$ and $\hat{A}_2$ are slowly varying operators, which satisfy the usual bosonic commutation relations: $[\hat{A}_i,\hat{A}_j^\dagger]=\delta_{ij}$. Then Hamiltonian (5) is represented in terms of the new boson operators as

$$H=H_A+H', \qquad (11)$$

where

$$H_A=\omega\hat{N}+\frac{\chi}{2}\hat{N}^2-\chi\hat{A}_1^\dagger\hat{A}_1\hat{A}_2^\dagger\hat{A}_2+\frac{q}{4}[3\hat{N}^2-(\hat{A}_1^\dagger\hat{A}_1-\hat{A}_2^\dagger\hat{A}_2)^2]$$
$$+g\Omega[\hat{A}_1^\dagger\hat{A}_1-\hat{A}_2^\dagger\hat{A}_2], \qquad (12)$$

$\omega=\omega_0-\frac{1}{2}(\chi+g)$. The total number is a conserved quantity, and the number operator is $\hat{N}=\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2=\hat{A}_1^\dagger\hat{A}_1+\hat{A}_2^\dagger\hat{A}_2$; H' denotes a nonresonant term written as

$$H'=\tfrac{1}{2}(\chi-q)(\hat{A}_1^{\dagger 2}\hat{A}_2^2e^{-i4gt}+\hat{A}_2^{\dagger 2}\hat{A}_1^2e^{i4gt}), \qquad (13)$$

and plays no essential role in the evolution of the measurable quantities specifying the macroscopic quantum phenomena of the two-condensates system [13]. In the experiments [5,12], $\Omega(\sim 6834\ \text{MHz})\gg\omega_z$ (~ 59 Hz) and q, $\chi\ll\omega_z$, so the condition $\Omega\gg(\chi-q)$ was achieved in the experiments [5,12]. Therefore, these nonresonant terms are fully negligible. Obviously, H_A is diagonal in the Fock space of $(\hat{A}_1,\hat{A}_2)$,

$$H_A|n,m)=E(n,m)|n,m), \qquad (14)$$

with eigenvalues

$$E(n,m)=\omega_0(n+m)+\tfrac{1}{4}(3g+2\chi)(n+m)^2-\tfrac{1}{4}q(n-m)^2$$
$$-\chi nm+g\Omega(n-m). \qquad (15)$$

The coherent states $|\alpha_1,\alpha_2\rangle$ and $|u_1,u_2)$, defined in Fock spaces of $(\hat{a}_1,\hat{a}_2)$ and $(\hat{A}_1,\hat{A}_2)$, respectively, are related to each other by

$$|\alpha_1,\alpha_2\rangle=|\frac{\alpha_1e^{i(\omega_{rf}/2)t}+\alpha_2e^{-i(\omega_{rf}/2)t}}{\sqrt{2}}$$
$$\times e^{igt},\frac{i(\alpha_1e^{i(\omega_{rf}/2)t}-\alpha_2e^{-i(\omega_{rf}/2)t})}{\sqrt{2}}e^{-igt}), \qquad (16)$$

Then the state for the amplitude of the two species of a condensate system at time t can be explicitly written as

$$|\Phi(t)\rangle=e^{-N/2}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}u_1^n(iu_2)^me^{-iE(n,m)t}|n,m), \qquad (17)$$

where $u_1=(\alpha_1+\alpha_2)/\sqrt{2}$, $u_2=(\alpha_1-\alpha_2)/\sqrt{2}$, and $N=|\alpha_1|^2+|\alpha_2|^2=|u_1|^2+|u_2|^2$.

In order to compare with the result in Ref. [5], we first calculate the condensate density of the second condensate $N_2(t)$ corresponding to the density in the spin state $|2\rangle$ in Ref. [5]:

$$N_2(t)=\langle\Phi(t)|a_2^\dagger a_2|\Phi(t)\rangle. \qquad (18)$$

Using transformation (10), we have

$$N_2(t)=\tfrac{1}{2}N-\text{Im}\,W(t), \qquad (19)$$

where $N=N_1+N_2$ and

$$W(t)=\tfrac{1}{2}e^{-A(t)}[\mathcal{R}(t)+i\mathcal{I}(t)], \qquad (20)$$

where

$$A(t)=N[1-\cos[(q-\chi)t]]+2|\alpha_1\alpha_2|\sin\omega_{rf}t\sin\varphi$$
$$\times\cos(q-\chi)t,$$

$$\mathcal{R}(t)=-(N_1-N_2)\cos\omega_{rf}t\cos 2\Theta(t)$$
$$+(N\sin\omega_{rf}t-2|\alpha_1\alpha_2|\sin\varphi)\sin 2\Theta(t), \qquad (21)$$

$$\mathcal{I}(t)=(N_1-N_2)\cos\omega_{rf}t\sin 2\Theta(t)$$
$$+(N\sin\omega_{rf}t-2|\alpha_1\alpha_2|\sin\varphi)\cos 2\Theta(t),$$

and

$$\Theta(t)=g(1-\Omega)t+|\alpha_1\alpha_2|\cos\omega_{rf}t\cos\varphi\sin(q-\chi)t.$$

$\alpha_1\alpha_2^*=|\alpha_1\alpha_2|e^{i\varphi}$, φ denotes the initial phase difference of the components of the condensates. Then the explicit expression of the second condensates density can be written as

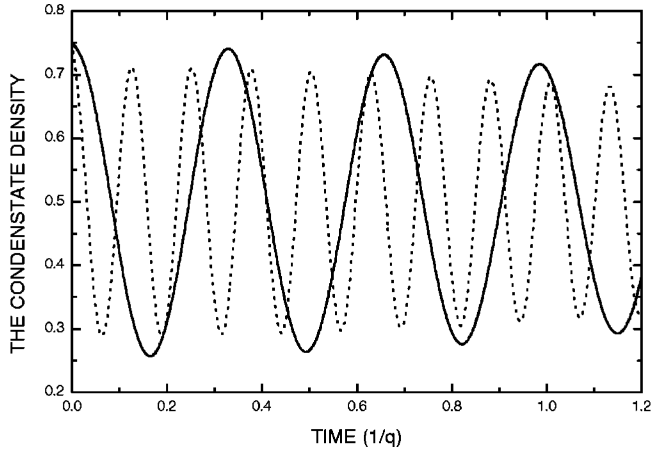

FIG. 1. The time evolution of the number density $N_2(t)$ with $\varphi=\pi/6$, $\omega_{rf}=0.001$, $\Omega=20\times2\pi$ (solid line), and $\Omega=50\times2\pi$ (dotted line).

$$N_2(t)=\tfrac{1}{2}N-\tfrac{1}{2}e^{-A(t)}\mathcal{I}(t). \tag{22}$$

From Eq. (22), we see that the initial phase is memorized in the density $N_2(t)$ during the time evolution. In addition, the evolution of the density manifests itself in the collapse and revival, as pointed out by several authors [13,4], and the Rabi frequency Ω affects the period of the collapse and revival of the density in a complex way. When the two-component BEC's initially possess equal particle numbers, as in Ref. [5], Eq. (22) can be simplified to

$$N_2(t)=\tfrac{1}{2}N-\tfrac{1}{2}e^{-A(t)}(N\sin\omega_{rf}t-2|\alpha_1\alpha_2|\sin\varphi)\cos 2\Theta(t). \tag{23}$$

To see how the Rabi frequency Ω of the coupling drive affects the density of the condensate, in Fig. 1 we plot the time evolution of $N_2(t)$ in the cases of various Rabi frequencies. Here we have assumed that $q=1$ and that the relative parameters $g=0.49$ and $\chi=0.5$ are measured in units of q and the time t is in $1/q$. Figure 1 shows that the form of the best fit $N_2(t)$ is almost close to a sine function, implying that in this case the coupling nearly drives a weak link between two condensates. It is also clear that with the increase of the Rabi frequency Ω, the amplitude and the period of oscillation of the condensate density decrease, which is in good agreement with the experimental observation reported in Refs. [5,7]. From Fig. 2, we see that the condensate density is highly sensitive to the initial value of the relative phase φ. When $\varphi=0$ or $\varphi=\pi$, there are no oscillations. The influence of the initial relative phase difference on the amplitude and period of the oscillations is similar as that of the Rabi frequency. In particular when $\varphi=\pi/6$, the oscillation is quite similar to the experimental results in Ref. [5].

With the help of the Eq. (22), $N_2(t)$, with various ratios N_2/N_1 is plotted in Fig. 3. The number density $N_2(t)$ seems insensitive to the value of the ratio N_2/N_1. This indicates that the coupling drive is not a factor of decoherence in such a system [14].

The interference between two BEC's may be studied from the expectation value of the density operator,

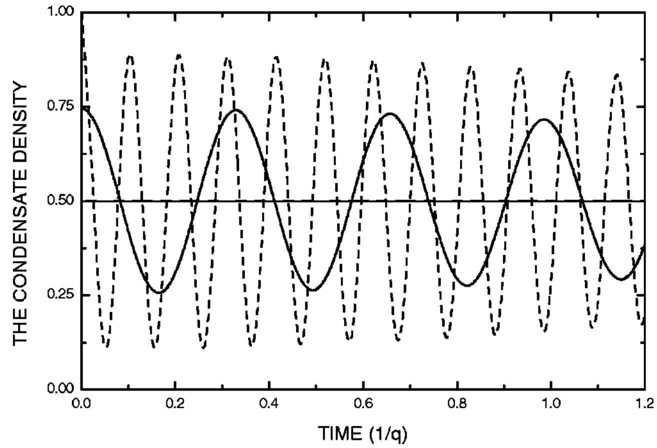

FIG. 2. $N_2(t)$ for various initial values for the relative phase $\varphi=0$ (straight solid line), $\pi/6$ (solid line), $\pi/2$ (dashed line), and $5\pi/6$ (dotted line), with $\omega_{rf}=0.001$ and $\Omega=20\times2\pi$.

$$I(t)=\langle\Phi(t)|\Psi^{\dagger}\Psi|\Phi(t)\rangle, \tag{24}$$

where $\Psi=(1/\sqrt{2})(\hat{a}_1+\hat{a}_2)$. With the help of the Eqs. (10), (17), (20), and (15), we have

$$I(t)=\tfrac{1}{2}N+\tfrac{1}{2}(N_1-N_2)\cos\omega_{rf}t+\tfrac{1}{2}e^{-A(t)}\sin\omega_{rf}t\mathcal{R}(t). \tag{25}$$

For the special case $N_1=N_2$, we have

$$I(t)=\tfrac{1}{2}N+\tfrac{1}{2}e^{-A(t)}(N\sin\omega_{rf}t-2|\alpha_1\alpha_2|\sin\varphi)\sin 2\Theta(t)\sin\omega_{rf}t. \tag{26}$$

Figure 4 displays the time evolution of the average density with various Rabi frequencies. From the stable interference patterns, we again see that the condensates retain a clear memory of their initial relative phase, consistent with the experimental result in Ref. [5]. Similarly, an increase in the Rabi frequency reduces the amplitude and period of the oscillation of the interference. The evolution of $I(t)$ is also plotted in Fig. 5, with various initial values of the relative

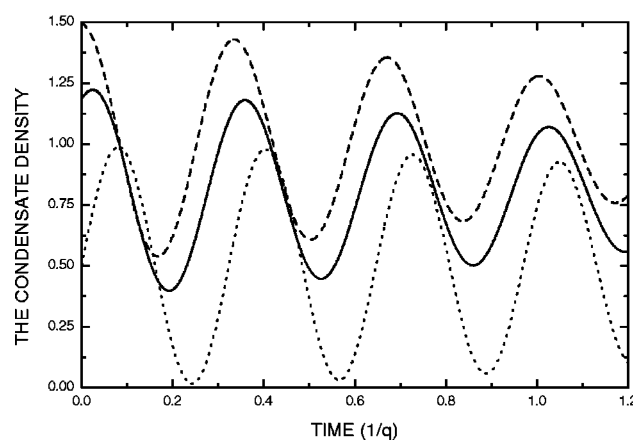

FIG. 3. $N_2(t)$ for various initial values of the ratio $N_2/N_1=0$ (dotted line), 0.6 (solid line), and 1 (dashed line), with $\varphi=\pi/6$, $\omega_{rf}=0.001$, and $\Omega=20\times2\pi$.

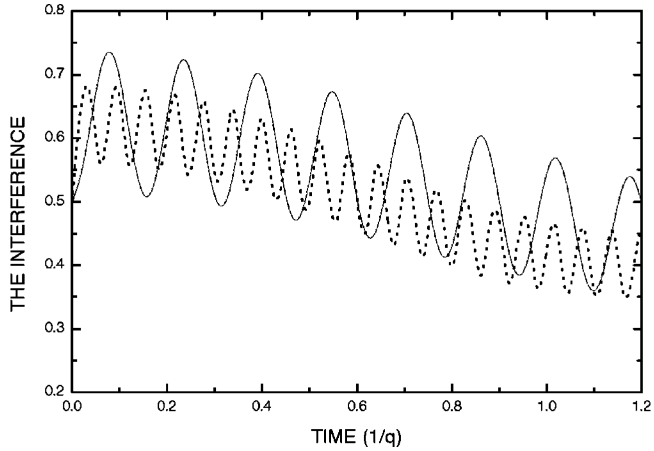

FIG. 4. The time evolution of the average density $I(t)$ with $\varphi=\pi/6$, $\omega_{rf}=0.001$, and $\Omega=20\times 2\pi$ (solid line), and $\Omega=50\times 2\pi$ (dotted line).

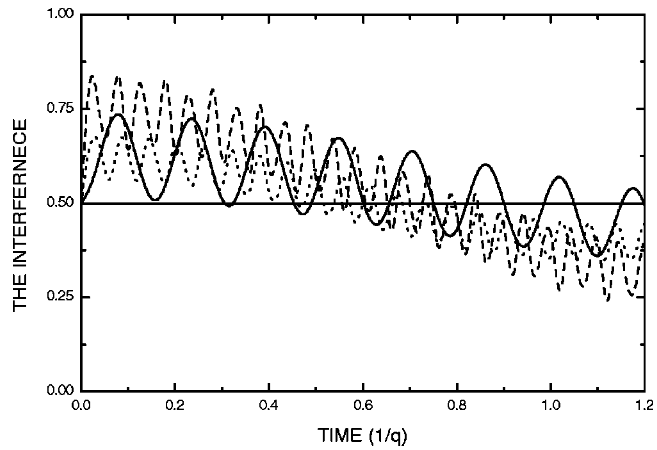

FIG. 5. $I(t)$ for various initial values for the relative phase $\varphi=0$ (straight line), $\pi/6$ (solid line), $\pi/2$ (dashed line), and $5\pi/6$ (dotted line) with $\varphi=\pi/6$, $\omega_{rf}=0.001$, and $\Omega=20\times 2\pi$.

phase. We find that in some special initial phase differences $(\varphi=0,\pi)$, there would be no oscillation at all.

In conclusion, making use of the time-dependent Bogoliubov transformation, we analytically obtained eigenstates for the amplitude of two-component BEC's, including the coupling drive, and further calculated the number density of the second condensate and the interference density. With the help of a numerical calculation, we found that the evolution of the number density of the individual condensate performs a collapse and revival with a complex period, and displays a good sinusoidal line which is similar to the current-phase relation of a superfluid ^{3}He-B weak link reported in Ref. [14]. The amplitude and period of the number density and the interference density decrease with increasing Rabi frequency Ω. Both the evolution of the number density and the interference density show that the coupling drive is not a factor of the decoherence [14]. The evolution of the condensate density is sensitive to the initial relative phase difference, and insensitive to the initial value of the number ratio N_2/N_1. In some special initial values of the relative phase $(\varphi=0,\pi)$, there would be no oscillations at all. Our theoretical result for $N_2(t)$, with $\varphi=\pi/6$, is in good agreement with the experimental observation in Ref. [5]. Our investigation may shed light on an understanding the coherent properties of condensates.

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[1] M.H. Anderson *et al.*, Science **269**, 198 (1995).
[2] K.B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[3] C.C. Bradley *et al.*, Phys. Rev. Lett. **75**, 1687 (1995); **79**, 1170 (1997).
[4] A.S. Parkins and D.F. Walls, Phys. Rep. **303**, 1-80 (1998).
[5] D.S. Hall *et al.*, Phys. Rev. Lett. **81**, 1543 (1998).
[6] A. J. Legget, in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S.Stringari (Cambridge University Press, Cambridge, 1995).
[7] M.R. Andrews *et al.*, Science **275**, 637 (1997).
[8] J. Javanainen, Phys. Rev. A **54**, R4629 (1996).
[9] A. Imamoglu and T.A.B. Kennedy, Phys. Rev. A **55**, R849 (1997).
[10] J. Ruostekoski and D.F. Walls, Phys. Rev. A **56**, 2996 (1997).
[11] Wu-Ming Liu, B. Wu, and Q. Niu, Phys. Rev. Lett. **84**, 2294 (2000).
[12] M.R. Matthews *et al.*, Phys. Rev. Lett. **81**, 243 (1998).
[13] L.M. Kuang *et al.*, Phys. Rev. A **61**, 013608 (2000).
[14] S. Backhaus *et al.*, Science **278**, 21 (1997).
[15] J.I. Cirac *et al.*, Phys. Rev. A **57**, 1208 (1998); M. Steel *et al.*, *ibid.* **57**, 2920 (1998).
[16] G.J. Milburn *et al.*, Phys. Rev. A **55**, 4318 (1997); E.M. Wright *et al.*, *ibid.* **56**, 591 (1997); M.J. Steel and D.F. Walls, *ibid.* **57**, 3805 (1998).