Superadditivity of quantum-correlating power

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We investigate the superadditivity property of quantum-correlating power (QCP), which is the power of generating quantum correlation by a local quantum channel. We prove that, when two local quantum channels are used in parallel, the QCP of the composed channel is no less than the sum of QCP of the two channels. For local channels with zero QCP, the superactivation of QCP is a fairly common effect, and it is proven to exist widely except for the trivial case where both of the channels are completely decohering channels or unitary operators. For general quantum channels, we show that the (not-so-common) additivity of QCP can be observed for the situation where a measuring-and-preparing channel is used together with a completely decohering channel.

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I. INTRODUCTION

Contrary to quantum entanglement, which is a monotone under local operations, general quantum correlations, such as quantum discord, can be created and increased by local operations [1–5]. This is understandable, as the local operations can turn some classical correlations into quantum ones. More and more evidence shows that quantum correlation is responsible for quantum-information processes, such as remote state preparation [6,7], entanglement distribution [8,9], quantum state discrimination [10], etc. The importance of quantum correlation also lies in its close connection to quantum entanglement [11–13]. The local creation of quantum correlation makes the preparation of a quantum-correlated state simple. Meanwhile, it provides a method to study the properties of both the quantum correlations and the local quantum channels. Generally, a local channel is able to create quantum correlation if and only if it is not a commutativity-preserving channel [2]. For the qubit case, a commutativity-preserving channel is either a completely decohering channel or a unital channel [1,2], while for the qutrit case, the set of commutativity-preserving channels reduces to completely decohering channels and isotropic channels [2].

When solving the problem of whether a channel can create quantum correlation, it is natural to investigate how much quantum correlation can be created locally. For this purpose, quantum-correlating power (QCP) was proposed as the maximum quantum correlation that can be created by a given local quantum channel [14]. QCP not only quantifies the amount of quantum correlation created by local operation but also serves as an inherent property of quantum channels. It is of interest to investigate the effect caused by using two channels together. We have given an example to indicate the superactivation of QCP of two zero-QCP channels in Ref. [14]. It implies that using two channels together is more efficient in creating quantum correlations than using them separately. It is straightforward to ask the following questions: What kind of local channel has the property of superactivation of QCP? Does superadditivity of QCP hold for general quantum channels? We remark that the additivity property is one of the most fundamental problems in quantum-information science; for example, it is shown that communication capacity using entangled inputs is superadditivity [15].

In this paper, we prove that when the two channels have zero QCP, the superactivation can be observed except when the two channels are both unitary operations or completely decohering channels. This means that the superactivation of QCP is a common phenomenon. We then prove that the answer to the first question is positive. When one of the channels has positive QCP, there are still situations where the QCP of the two channels is additive. The QCP of the bichannel which is composed of a measuring-and-preparing (MP) channel and a completely decohering channel equals that of the MP channel. We consider the genuine quantum correlation to be responsible for the superadditivity of QCP.

II. SUPERACTIVATION OF QUANTUM-CORRELATING POWER

Let us briefly recall the definition of QCP [14] for local channels. The quantum-correlating power of a quantum channel Λ is defined as

$$\mathcal{Q}(\Lambda) = \max_{\rho\in\mathcal{C}_0} Q(\Lambda\otimes I(\rho)). \tag{1}$$

Here $\mathcal{C}_0$ is a set of classical-quantum states, which can be written as [16]

$$\mathcal{C}_0 = \left\{\rho|\rho = \sum_i q_i \Pi^A_{\alpha_i}\otimes\rho^B_i\right\}, \tag{2}$$

and Q is a measure of quantum correlation satisfying the following three conditions: (a) $Q(\rho)=0$ iff $\rho\in\mathcal{C}_0$; (b) $Q(U\rho U^\dagger)=Q(\rho)$, where U is a local unitary operator on A or B; and (c) $Q[I\otimes\Lambda_B(\rho)]\leqslant Q(\rho)$.

In Ref. [14], we have given an example to show the effect of superactivation for QCP of phase-damping qubit channels $\Lambda^{\rm PD}$. Precisely, we consider the input state $\rho=\frac{1}{4}\sum_{i,j}|\psi_{ij}\rangle_{AA'}\langle\psi_{ij}|\otimes|ij\rangle_{BB'}\langle ij|$, with $|\psi_{00}\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $|\psi_{11}\rangle=\frac{1}{\sqrt{2}}(|0+\rangle+|1-\rangle)$, $|\psi_{01}\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$, and $|\psi_{10}\rangle=\frac{1}{\sqrt{2}}(|0-\rangle-|1+\rangle)$. Here, the qubits A and A' belong to Alice while B and B' belong to Bob. Hence, Alice and Bob are classically correlated initially. Now let the composed channel $\Lambda^{\rm PD}_A\otimes\Lambda^{\rm PD}_{A'}$ act on the qubits A and A'. The resulting state has nonzero quantum

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correlation because $[\Lambda^{\rm PD}\otimes\Lambda^{\rm PD}(\psi_{00}),\Lambda^{\rm PD}\otimes\Lambda^{\rm PD}(\psi_{11})]=\frac{1}{8}\tilde{i}p\sqrt{1-p}(\sigma^y\otimes\sigma^z+\sigma_z\otimes\sigma^y)\neq 0$. Therefore, the QCP of the composed channel $\Lambda^{\rm PD}\otimes\Lambda^{\rm PD}$ is nonzero.

Then what kind of zero-QCP channel has the property of superactivation of QCP? Obviously, if the two channels are both completely decohering channels, the composed channel is still a completely decohering channel, which is not able to create quantum correlation. Similarly, for the case where the two channels are both unitary operators, the composed channel is a unitary operator too and thus has zero QCP. In the following, we prove that these are the only two situations where superactivation of QCP is not observed.

Theorem 1. For two zero-QCP channels Λ_1 and Λ_2, the QCP of the composite channel $\Lambda_1\otimes\Lambda_2$ is nonzero except that both Λ_1 and Λ_2 are completely decohering channels or unitary channels.

Proof. From Ref. [2], the QCP of channel $\Lambda_1\otimes\Lambda_2$ is zero if and only if $\Lambda_1\otimes\Lambda_2$ is a commutativity-preserving channel. It means that, for any commutative two-particle states ξ_1 and ξ_2, we have

$$\left[\Lambda_1^A\otimes\Lambda_2^{A'}(\xi_1),\Lambda_1^A\otimes\Lambda_2^{A'}(\xi_2)\right]=0. \tag{3}$$

Obviously, when both Λ_1 and Λ_2 are completely decohering channels or unitary channels, the channel $\Lambda_1\otimes\Lambda_2$ is also a commutativity-preserving channel, and Eq. (3) holds. Otherwise, let Λ_1 not be a completely decohering channel and Λ_2 not be a unitary channel. Now we choose $\xi_1=|0\rangle_A\langle 0|\otimes|\psi\rangle_{A'}\langle\psi|$ and $\xi_2=|\theta\rangle_A\langle\theta|\otimes|\psi^\perp\rangle_{A'}\langle\psi^\perp|$, where $\langle\psi|\psi^\perp\rangle=0$ and $|\theta\rangle$ is an arbitrary single-particle state. Then Eq. (3) is equivalent to

$$[\Lambda_1(|0\rangle\langle 0|),\Lambda_1(\theta)]\otimes\Lambda_2(\psi)\Lambda_2(\psi^\perp)=0. \tag{4}$$

Here we label ψ as the density matrix of the pure state $|\psi\rangle$ and similar for $\psi^\perp$ and θ. Since channel Λ_1 is not a completely decohering channel, we can always find a single-particle state $|\theta\rangle$ such that $[\Lambda_1(|0\rangle\langle 0|),\Lambda_1(\theta)]\neq 0$. Meanwhile, when Λ_2 is not a unitary channel, there exist two pure orthogonal single-particle states ψ and $\psi^\perp$ such that $\Lambda_2(\psi)\Lambda_2(\psi^\perp)\neq 0$. Therefore, Eq. (4) is violated. This completes the proof of Theorem 1.

In the above discussion, superactivation of QCP is due to the nonclassicality of particle A. An extreme example is that $\Lambda_1=I$ is an identity channel while Λ_2 is a completely depolarizing channel, which is equivalent to the "trace-out" operation. Now we start with the state $\rho_{AB}\otimes|0\rangle_{A'}\langle 0|$, where $\rho_{AB}\in\mathcal{C}_0$. After a two-particle unitary operator $U_{AA'}$ on A and A', which does not create quantum correlation on the left, the channel $I\otimes\Lambda_2$ is applied. This process can be expressed as

$$\rho_{\rm out}=\Lambda_A^U(\rho_{AB})\otimes\frac{I_{A'}}{2}, \tag{5}$$

where $\Lambda_A^U(\rho_AB)={\rm Tr}_{A'}(U_{AA'}\rho_{AB}\otimes|0\rangle_{A'}\langle 0|U_{AA'}^\dagger)$. Therefore, the superactivation of the two channels $\Lambda_1=I$ and Λ_2 is in fact local creation of quantum correlation by the channel Λ_A^U.

We then focus on situations where no pairwise quantum correlations are induced between A and $\tilde{B}$ or between A' and $\tilde{B}$, where $\tilde{B}$ is the system on Bob's side. The mechanism for superactivation of QCP under this condition is totally different from that for the local creation of quantum correlation. In this case, the two states ξ_1 and ξ_2 used for checking nonzero QCP of the channel $\Lambda_1\otimes\Lambda_2$ as in Eq. (3) should satisfy

$$\left[\xi_1^A,\xi_2^A\right]=0,\quad\left[\xi_1^{A'},\xi_2^{A'}\right]=0. \tag{6}$$

Here $\xi_i^{A(A')}={\rm Tr}_{A'(A)}\xi_i$ are reduced density matrices. In the following, we see that, even limited to the situation where no pairwise quantum correlation is induced, the superactivation of QCP can still be observed for most of the channels.

We first discuss the situation where Λ_1 and Λ_2 are the qubit channels. According to Refs. [1,2], Λ has zero QCP only when it is a completely decohering channel, $\Lambda^{\rm CD}$, or a unital channel, $\Lambda^{\rm I}$. We focus on the superactivation of QCP for two unital channels. A unital channel is defined as a channel which keeps the identity operator invariant:

$$\Lambda^{\rm I}\equiv\{\Lambda:\Lambda(I)=I\}. \tag{7}$$

It has been proven that any unital channel of a qubit is unitarily equivalent to a Pauli channel [17] $\Lambda^{\rm I}(\cdot)=v\Lambda^{\rm Pauli}[u^\dagger(\cdot)u]v^\dagger$. Here the Kraus operators of a Pauli channel are proportional to Pauli matrices:

$$\Lambda^{\rm Pauli}(\cdot)=\sum_{i=0}^{3}\lambda_i\sigma_i(\cdot)\sigma_i, \tag{8}$$

where $\sigma_0=I$, $\sigma_{1,2,3}$ are the three Pauli matrices, $\lambda_i\geqslant 0$, $\lambda_0\geqslant\lambda_{1,2,3}$, and $\sum_{i=0}^3\lambda_i=1$. Therefore, it is adequate to consider the superactivation of QCP for Pauli channels.

Theorem 2. When limited to the case where no pairwise quantum correlation is induced, the superactivation of QCP can be observed for unital qubit channels which are not one of the following cases: (a) both channels are identical isotropic channels, and (b) one of the channels is a completely depolarizing channel.

Proof. For case (b), consider that channel Λ_2 is a completely depolarizing channel. The left-hand side of Eq. (3) is equal to $[\Lambda_1(\xi_1^A),\Lambda_1(\xi_2^A)]\otimes I^{A'}/2$, which vanishes under the constraint of Eq. (6) and the superactivation of QCP cannot be observed. In the following, we focus on the cases where none of the two channels are completely depolarizing channels. ξ_1 and ξ_2 in Eq. (3) is chosen as $\xi_i=|\Phi_i\rangle\langle\Phi_i|$, where $|\Phi_1\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$ and $|\Phi_2\rangle=(|0+\rangle+|1-\rangle)/\sqrt{2}$. In Pauli presentation, we have

$$\begin{aligned}\xi_1&=\tfrac{1}{4}(I\otimes I+\sigma_1\otimes\sigma_1-\sigma_2\otimes\sigma_2+\sigma_3\otimes\sigma_3),\\ \xi_2&=\tfrac{1}{4}(I\otimes I+\sigma_1\otimes\sigma_3+\sigma_2\otimes\sigma_2+\sigma_3\otimes\sigma_1).\end{aligned} \tag{9}$$

For Λ_1 being a Pauli channel, we have $\Lambda_1(\sigma_i)=a_i\sigma_i$, with $a_i=\lambda_0^{(1)}+2\lambda_i^{(1)}-\sum_i\lambda_i^{(1)}$, $i=1,2,3$. Similarly, $\Lambda_2(\sigma_i)=b_i\sigma_i$. Consequently, $\Lambda_1\otimes\Lambda_2(\xi_1)=\frac{1}{4}(I\otimes I+a_1b_1\sigma_1\otimes\sigma_1-a_2b_2\sigma_2\otimes\sigma_2+a_3b_3\sigma_3\otimes\sigma_3)$ and $\Lambda\otimes\Lambda(\xi_2)=\frac{1}{4}(I\otimes I+a_1b_3\sigma_1\otimes\sigma_3+a_2b_2\sigma_2\otimes\sigma_2+a_3b_1\sigma_3\otimes\sigma_1)$. Therefore, Eq. (3) means that

$$a_1a_3\left(b_1^2-b_3^2\right)=0,\quad b_1b_3\left(a_1^2-a_3^2\right)=0. \tag{10}$$

By alternating the subscripts 1, 2, and 3 in Eq. (9) and adopting Eq. (3), we have

$$\begin{aligned}&a_2a_3\left(b_2^2-b_3^2\right)=0,b_2b_3\left(a_2^2-a_3^2\right)=0,\\ &a_1a_2\left(b_1^2-b_2^2\right)=0,b_1b_2\left(a_1^2-a_2^2\right)=0.\end{aligned} \tag{11}$$

This means that the superactivation of QCP for channels violating Eqs. (10) or (11) can be detected by the states ξ_1 and ξ_2 in the form of Eq. (9) or those obtained by alternating the subscripts 1,2,3 in Eq. (9). Then we are left with the following situations.

(1) $|a_2| = |a_3| = a > 0$, $|b_2| = |b_3| = b > 0$, and $a_1 = b_1 = 0$. Noticing that conditions $a_1 = 0$ and $\lambda_0 \geqslant \lambda_{1,2,3}$ lead to $a_{2,3} \geqslant 0$, we have $a_2 = a_3 = a > 0$ and $b_2 = b_3 = b > 0$. Λ_1 and Λ_2 are both projecting-and-depolarizing channels Λ^{Dp}, which take a state

$$\zeta = (I + \vec{r}\cdot\vec{\sigma})/2, \tag{12}$$

to $\Lambda_\chi^{\mathrm{Dp}}(\zeta) = [I + \chi(r_2\sigma_2 + r_3\sigma_3)]/2$, where $\chi = a,b$. In other words, the channel projects the Bloch vector $\vec{r}$ onto the x-y plane and then shortens it, $\Lambda_\chi^{\mathrm{Dp}} \equiv \Lambda_\chi^{\mathrm{D}} \circ \Lambda^{p}$.

(2) $|a_2| = |a_3| = a > 0$, $b_2 = b_3 = a_1 = 0$, and $b_1 \neq 0$. Discussions similar to those in Case (1) give that $a_2 = a_3 = a > 0$ and $b_1 = b > 0$. $\Lambda_1 = \Lambda_a^{\mathrm{Dp}}$ while $\Lambda_2 = \Lambda_b^{\mathrm{DPD}} \equiv \Lambda_b^{\mathrm{D}} \circ \Lambda^{\mathrm{PD}}$ is equivalent to a completely dephasing channel Λ^{PD} followed by a depolarizing channel Λ_b^{D}.

(3) $|a_1| = |a_2| = |a_3| > 0$ and $|b_1| = |b_2| = |b_3| > 0$. If $a_1 = -a_2 = a_3 \neq 0$, we have $\lambda_0 = \lambda_1 = \lambda_3 > \lambda_2$. This channel is equivalent to the isotropic channel with $\lambda_1 = \lambda_2 = \lambda_3 > \lambda_0$ by a unitary operator σ_2. Therefore, this case is equivalent to $a_1 = a_2 = a_3 = a \neq 0$ and $b_1 = b_2 = b_3 = b \neq 0$. Λ_1 and Λ_2 are isotropic channels.

(4) $b_1 = b_2 = b_3 = 0$. Channel Λ_2 is a completely depolarizing channel, which we have already considered.

Since two completely decohering channels do not have the property of superactivation of QCP, let Λ_1 not be a completely decohering channel. Cases (1) and (2) include the channels obtained by alternating the subscripts 1, 2, and 3. In the following, we derive the commutative states ξ_1 and ξ_2 satisfying Eq. (6) to detect the superactivation for these cases, and we prove that such states do not exist for Case (3) with $a_1 = b_1$.

Writing a two-qubit state in the form

$$\xi_k = \frac{1}{4}\left[I\otimes I + \sum_{i=1}^{3} r_i^k \sigma_i \otimes I + \sum_{i=1}^{3} s_i^k I \otimes \sigma_i + \sum_{i,j=0}^{3} T_{ij}^k \sigma_i \otimes \sigma_j \right], \tag{13}$$

where $k = 1,2$, we can present ξ_k as $\xi_k = \{\vec{r}^k, \vec{s}^k, \hat{T}^k\}$. It is worth noting that the commutation $[\xi_1,\xi_2]$ can also be written as the Bloch decomposition

$$[\xi_1,\xi_2] = \frac{\tilde{i}}{16}\left[\sum_{i=1}^{3} \alpha_i \sigma_i \otimes I + \sum_{i=1}^{3} \beta_i I \otimes \sigma_i + \sum_{i,j=0}^{3} \Gamma_{ij}\sigma_i\otimes\sigma_j \right], \tag{14}$$

where $\tilde{i}$ is the imaginary unit. By using the commutation for Pauli matrices, we have $[\sigma_i \otimes I, \sigma_{i'} \otimes \sigma_{j'}] = [\sigma_i,\sigma_{i'}] \otimes \sigma_{j'}$, $[\sigma_i \otimes I, \sigma_i' \otimes I] = [\sigma_i,\sigma_{i'}] \otimes I$, and $[\sigma_i \otimes \sigma_j, \sigma_{i'} \otimes \sigma_{j'}] = [\sigma_i,\sigma_{i'}] \otimes I\delta_{jj'} + I \otimes [\sigma_j,\sigma_{j'}]\delta_{ii'}$, where $\delta_{jj'}$ is the Kroneker δ. Therefore,

$$\begin{aligned} \Gamma_{ij} &= \left(\vec{r}^1 \times \vec{T}_{r_j}^2 - \vec{r}^2 \times \vec{T}_{r_j}^1\right)_i + \left(\vec{s}^1 \times \vec{T}_{s_i}^2 - \vec{s}^2 \times \vec{T}_{s_i}^1\right)_j, \\ \vec{\alpha} &= \vec{r}^1 \times \vec{r}^2 + \sum_j \vec{T}_{r_j}^1 \times \vec{T}_{r_j}^2, \\ \vec{\beta} &= \vec{s}^1 \times \vec{s}^2 + \sum_i \vec{T}_{s_i}^1 \times \vec{T}_{s_i}^2, \end{aligned} \tag{15}$$

where $\vec{T}_{r_j}^k = \{T_{1j}^k, T_{2j}^k, T_{3j}^k\}$ and $\vec{T}_{s_i}^k = \{T_{i1}^k, T_{i2}^k, T_{i3}^k\}$. Then $[\xi_1,\xi_2] = 0$ is equivalent to

$$\vec{\alpha} = \vec{\beta} = 0, \quad \hat{\Gamma} = 0. \tag{16}$$

Meanwhile, $[\xi_1^A, \xi_2^A] = 0$ and $[\xi_1^{A'}, \xi_2^{A'}] = 0$ give that

$$\vec{r}^1 \times \vec{r}^2 = 0, \quad \vec{s}^1 \times \vec{s}^2 = 0. \tag{17}$$

For Cases (1) and (2), as well as for Case (3) with $a \neq b$, we can find states ξ_1 and ξ_2 satisfying Eqs. (16) and (17) such that Eq. (3) is violated. Precisely, for Case (1) we chose ξ_k to be

$$\begin{aligned} \vec{r}^1 &= \vec{s}^1 = \{0,0,r\}, \vec{r}^2 = \vec{s}^2 = \{0,0,nr\}, \\ \hat{T}^1 &= \hat{T}^2 = \mathrm{diag}\{t,t,t\}. \end{aligned} \tag{18}$$

Direct calculation leads to $[\Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{Dp}}(\xi_1), \Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{Dp}}(\xi_2)] = \tilde{i}abrt(1-n)(b\sigma_2 \otimes \sigma_1 + a\sigma_1 \otimes \sigma_2) \neq 0$. For Case (2), we chose ξ_1 and ξ_2 to be

$$\begin{aligned} \vec{s}^1 &= \vec{r}^2 = \{0,r,r\}, \vec{r}^1 = \vec{s}^2 = 0, \\ \hat{T}^1 &= (\hat{T}^2)^{\mathrm{T}} = \{0,0,0; t,0,0; -t,0,0\}, \end{aligned} \tag{19}$$

and then $[\Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{DPD}}(\xi_1), \Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{DPD}}(\xi_2)] = 2\tilde{i}abrt\sigma_1 \otimes \sigma_1 \neq 0$. For Case (3) with $a \neq b$, we chose ξ_1 and ξ_2 as in Eq. (19), and then $[\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_b^{\mathrm{Iso}}(\xi_1), \Lambda_a^{\mathrm{Iso}} \otimes \Lambda_b^{\mathrm{Iso}}(\xi_2)] = 2\tilde{i}abrt(a-b)\sigma_1 \otimes \sigma_1 \neq 0$.

Now we only need to prove that for two identical isotropic channels, the superactivation of QCP cannot be detected when under the constraint that no pairwise quantum correlation is induced. From the property of the isotropic channels

$$\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_k) = \{a\vec{r}^k, a\vec{s}^k, a^2\hat{T}^k\}, \tag{20}$$

and consequently, the commutation of the output states can be written as $[\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_1), \Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_2)] = \{\vec{\alpha}^{\mathrm{Iso}}, \vec{\beta}^{\mathrm{Iso}}, \hat{\Gamma}^{\mathrm{Iso}}\}$, where $\vec{\alpha}^{\mathrm{Iso}} = a^2\vec{r}^1 \times \vec{r}^2 + a^4 \sum_j \vec{T}_{r_j}^1 \times \vec{T}_{r_j}^2$, $\vec{\beta}^{\mathrm{Iso}} = a^2\vec{s}^1 \times \vec{s}^2 + a^4\sum_i \vec{T}_{s_i}^1 \times \vec{T}_{s_i}^2$, and $\hat{\Gamma}^{\mathrm{Iso}} = a^3\hat{\Gamma}$. Therefore, Eqs. (16) and (17) imply $\vec{\alpha}^{\mathrm{Iso}} = \vec{\beta}^{\mathrm{Iso}} = 0$ and $\hat{\Gamma}^{\mathrm{Iso}} = 0$. This means that for two identical isotropic channels, the superactivation of QCP cannot be detected under the constraint that no pairwise quantum correlation is induced. This completes the proof of Theorem 2.

It is quite interesting that a completely dephasing channel can activate the QCP of a depolarizing channel, even under the constraint that no pairwise quantum correlation is induced. More precisely, we consider the initial classical-classical state $\rho = \sum_{i=0}^{3} |\Phi_i\rangle_{AA'}\langle\Phi_i| \otimes |i\rangle_{\tilde{B}}\langle i|$, where $|\Phi_0\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, $|\Phi_3\rangle = (|{-}0\rangle - |{+}1\rangle)/\sqrt{2}$, and $|\Phi_{1,2}\rangle$ are the same as in Eq. (9). Let Λ_A be a depolarizing channel with $\lambda_0 = (1+3a)/4$ and $\lambda_1 = \lambda_2 = \lambda_3 = (1-a)/4$ as in Eq. (8), and let $\Lambda_{A'}$ be a completely dephasing channel with $\lambda_0 = \lambda_3 = 1/2$ and $\lambda_1 = \lambda_2 = 0$. The output state $\Lambda_A \otimes \Lambda_{A'} \otimes I_{\tilde{B}}(\rho)$ has a positive quantum correlation since $[\Lambda_A \otimes \Lambda_{A'}(\Phi_1),\Lambda_A \otimes \Lambda_{A'}(\Phi_2)] = -\frac{\tilde{i}}{4}a^2\sigma_2 \otimes \sigma_0$, which is nonzero for $a \neq 0$. Let us look more closely at the correlation structure of the input state ρ. Clearly, the pairwise quantum correlation between any two particles is zero. However, the genuine quantum correlation is nonzero. This is because of the entanglement between the qubit A and the composite system $A'\tilde{B}$. Measuring $\tilde{B}$ on basis $\{|i\rangle\}$ and locally operating A and A' can distill a singlet of qubits A and A'. Therefore, the superactivation of QCP can be understood as transferring the genuine quantum correlation to the quantum correlation between the bipartition AA' and $\tilde{B}$. Notice that the generated quantum correlation is still genuine correlation, because the pairwise quantum correlation is still zero in the output state.

Now we have studied the case where Λ_1 and Λ_2 are high-dimension channels. We briefly show that Theorem 2 does not hold for the general situation where Λ_1 and Λ_2 are qu*d*it channels with $d \geqslant 3$. A qu*d*it channel (with $d \geqslant 3$) has zero QCP if and only if it is either a completely decohering channel or an isotropic channel. It is obvious that when Λ_2 is a completely depolarizing channel, Eq. (3) holds for any commutative states ξ_1 and ξ_2 which satisfy Eq. (6). However, for Case (a) in Theorem 2 where both Λ_1 and Λ_2 are identical isotropic channels, Eq. (3) can be violated by some commutative states ξ_1 and ξ_2 which satisfy Eq. (6). An equivalent statement for Case (a) in Theorem 2 is that for two commutative two-qubit states ξ_1 and ξ_2 which satisfy Eq. (6), the following equation holds

$$\left[\xi_1,\xi_2^A + \xi_2^{A'}\right] = \left[\xi_2,\xi_1^A + \xi_1^{A'}\right]. \tag{21}$$

The equivalence is obvious when we notice that $\Lambda_a^{\text{Iso}} \otimes \Lambda_a^{\text{Iso}}(\xi_k) = a^2\xi_k + a(1-a)(\xi_k^A + \xi_k^{A'})/2 + I^{AA'}/4$. However, Eq. (21) does not always hold for A and A' being qu*d*its with $d \geqslant 3$. For example, consider $d = 4$ and that the two commutative two-qu*d*it states are $\xi_1 = [I + T_1(\sigma_1 \otimes \sigma_2)_A \otimes (\sigma_3 \otimes \sigma_1)_{A'} + T_2(\sigma_3 \otimes \sigma_1)_A \otimes (\sigma_3 \otimes \sigma_2)_{A'}]/16$ and $\xi_2 = [I + T_1 I^A \otimes (\sigma_1 \otimes \sigma_2)_{A'} + T_2(\sigma_2 \otimes \sigma_3)_A \otimes (\sigma_1 \otimes \sigma_1)_{A'}]/16$, where $T_{1,2} \neq 0$. Then Eq. (21) is violated, since the left-hand side of Eq. (21) is proportional to $\tilde{i}T_1T_2(\sigma_3 \otimes \sigma_1) \otimes (\sigma_2 \otimes \sigma_0) \neq 0$ while the right-hand side equals zero. Therefore, Case (a) in Theorem 2 does not hold for channels of higher dimensions. This means that the QCP high-dimension channels are more easily superactivated.

III. SUPERADDITIVITY OF QCP FOR GENERAL CHANNELS

Here we prove the superadditivity of QCP for general channels. In the following, we only discuss the problem in the regime in which the quantum correlation Q in Eq. (1) is quantum discord [18,19], which is defined as

$$\delta_{B|A}(\rho) = \min_{\{F_i^A\}} S\left(\rho_{B|\{F_i^A\}}\right) - S_{B|A}(\rho), \tag{22}$$

where $S_{B|A}(\rho) = S(\rho) - S(\rho_A)$, with $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ being the conditional entropy; $\{F_i^A\}$ is a positive operator-valued measure (POVM) on qudit A; and $S(\rho_{B|\{F_i^A\}}) = \sum_i p_i S(\rho_{B|F_i^A})$, with $p_i = \text{Tr}(\rho F_i^A)$ and $\rho_{B|F_i^A} = \text{Tr}_A(\rho F_i^A)/p_i$ being the average entropy of B after the measurement.

Theorem 3. When two channels Λ_1 and Λ_2 are used in parallel, the QCP of the composite channel $\Lambda_1 \otimes \Lambda_2$ is no less than the sum of the QCP for the two channels:

$$\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) \geqslant \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2). \tag{23}$$

Proof. Let ρ_1 and ρ_2 be the optimal input states of Λ_1 and Λ_2, respectively, then we have $\mathcal{Q}(\Lambda_i) = D(\rho_i')$ with $\rho_i' \equiv \Lambda_i \otimes I(\rho_i)$, $i = 1,2$. As proved in Ref. [20], the classical correlation is additive for separable states $J(\rho \otimes \sigma) = J(\rho) + J(\sigma)$ when ρ is a separable state. Obviously, ρ_i' are separable states. Therefore, we have

$$\delta(\rho_1' \otimes \rho_2') = \delta(\rho_1') + \delta(\rho_2'). \tag{24}$$

Since $\rho_1 \otimes \rho_2$ may not be the optimal input state for channel $\Lambda_1 \otimes \Lambda_2$, from the definition of QCP, $\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) \geqslant \delta(\rho_1' \otimes \rho_2') = \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2)$. This completes the proof of Theorem 3.

From the discussions in the last section, we observe that $\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) > \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2)$ is quite common. Therefore, we ask the following question: are there situations where the QCP is additive for channels with positive QCP? We give a positive answer to this question by providing a class of channels whose QCP is additive.

Here we define the measuring-and-preparing (MP) channel as the operation which measures on a fixed orthogonal basis and then prepares the qubit to predefined states conditioned on the measurement results. All MP channels are unitarily equivalent to

$$\Lambda^{\text{MP}}(\rho) = \sum_{i=0}^{d-1}\langle i|\rho|i\rangle\eta_i, \tag{25}$$

where η_i are quantum states. Belonging to the set of MP channels are the completely decohering channels, as well as the single-qubit channel with maximum QCP, whose Kraus operators are $E_0^{\text{M}} = |0\rangle\langle 0|$ and $E_1^{\text{M}} = |+\rangle\langle 1|$. For any MP channels in the form of Eq. (25), the optimal input state to reach the maximum quantum discord in the output state is

$$\rho^{\text{MP}} = \sum_{i=0}^{d-1} p_i|i\rangle\langle i| \otimes |i\rangle\langle i|. \tag{26}$$

The reason is as follows. Writing the general form of an optimal input state $\rho = \sum_{i=0}^{d-1} q_i|\phi_i\rangle\langle\phi_i| \otimes |i\rangle\langle i|$, we have the corresponding output state

$$\rho' = \sum_{i=0}^{d-1} p_i\eta_i \otimes \rho_i, \tag{27}$$

where $p_i = \sum_j q_j|\langle i|\phi_j\rangle|^2$ and $\rho_i = \sum_j q_j|\langle i|\phi_j\rangle|^2|j\rangle\langle j|/p_i$. Equation (27) can be obtained from $\Lambda^{\text{MP}} \otimes I(\rho^{\text{MP}})$ by local operation on B, which cannot increase the quantum discord. Therefore, the optimal input state should be in the form of Eq. (26).

Theorem 4. When a MP channel Λ^{MP} and a completely decohering channel Λ^{CD} are used in parallel, the QCP of the composed channel is equal to that of Λ^{MP}:

$$\mathcal{Q}(\Lambda^{\text{MP}} \otimes \Lambda^{\text{CD}}) = \mathcal{Q}(\Lambda^{\text{MP}}). \tag{28}$$

Proof. Consider the general form of an optimal input state,

$$\rho = \sum_{i,j=0}^{d-1} p_{ij} \Pi_{\phi_{ij}}^{AA'} \otimes \Pi_{\psi_{ij}}^{BB'}, \tag{29}$$

where $|\phi_{ij}\rangle = U|ij\rangle$ and $|\psi_{ij}\rangle = |ij\rangle$. Without loss of generality, we assume that the basis of the completely decohering channel is $\{|i\rangle\}$ and that the MP channel is in the form of Eq. (25). The QCP of the channel $\Lambda^{\rm MP}$ is just the quantum discord in state $\Lambda^{\rm MP} \otimes I(\rho^{\rm MP})$. Notice that

$$\Lambda_{A'}^{\rm CD}\left(\Pi_{\phi_{ij}}^{AA'}\right) = \sum_{k=0}^{d-1} q_{ij}^{k} \rho_{ij}^{(k)A} \otimes |k\rangle_{A'}\langle k|, \tag{30}$$

and the state $\rho' = \Lambda_{A'}^{\rm CD}(\rho)$ is of the form

$$\rho' = \sum_{k=0}^{d-1} |k\rangle_{A'}\langle k| \otimes \sum_{i,j=0}^{1} p_{ij} q_{ij}^{k} \rho_{ij}^{(k)A} \otimes \Pi_{\beta_{ij}}^{BB'}. \tag{31}$$

The output state $\tilde{\rho} = \Lambda_{A}^{\rm MP} \otimes \Lambda_{A'}^{\rm CD}(\rho)$ is then

$$\tilde{\rho} = \sum_{k=0}^{d-1} r_k |k\rangle_{A'}\langle k| \otimes \rho_k, \tag{32}$$

where $r_k = \sum_{i,j=0}^{d-1} p_{ij} q_{ij}^{k}$, $\rho_k = \sum_l \eta_l^A \otimes \xi_{lk}^{BB'}$, and $\xi_{lk}^{BB'} = \sum_{i,j=0}^{1} p_{ij} q_{ij}^{l} \langle k|\rho_{ij}^{(l)}|k\rangle \Pi_{\beta_{ij}}^{BB'}/r_k$. Obviously, $\delta_{BB'|A}(\rho_k) \leqslant \mathcal{Q}(\Lambda^{\rm MP})$ for $k = 1,2,\ldots,d-1$. We prove in the following that, for states in the form of Eq. (32), the quantum discord of $\tilde{\rho}$ is lower bounded by the weighted average quantum discord of ρ_k:

$$\delta_{BB'|AA'}(\tilde{\rho}) \leqslant \sum_{k=0}^{d-1} r_k \delta_{BB'|A}(\rho_k). \tag{33}$$

Suppose the optimal POVM for ρ_k is $\{F_k^{(i)}\}_{i=0}^{N_k-1}$. By building a POVM on qubits A and A' as $\{G^{(l)}\}_{l=0}^{\sum_k N_k-1} = \{|k\rangle\langle k| \otimes F_k^{(i)}\}_{k=0,\ldots,d-1}^{i=0,\ldots,N_k-1}$, we have

$$\begin{aligned} S\left(\tilde{\rho}_{BB'|\{F_{AA'}^{(l)}\}}\right) &= \sum_l \tilde{p}_l S\left(\frac{{\rm Tr}_{AA'}\left(G_{AA'}^{(l)}\tilde{\rho}\right)}{\tilde{p}_l}\right) \\ &= \sum_{ik} r_k \tilde{q}_{ki} S\left(\frac{{\rm Tr}_{A'}\left(F_{k,A'}^{(i)}\rho_k\right)}{\tilde{q}_{ki}}\right) \\ &= \sum_{k=0}^{d-1} r_k S\left(\rho_{k,BB'|\{F_k^{(i)}\}}\right). \end{aligned} \tag{34}$$

Meanwhile, direct calculation leads to $S(\tilde{\rho}_{BB'|AA'}) = \sum_{k=0}^{d-1} r_k S(\rho_{k,BB'|AA'})$. Consequently, $S(\tilde{\rho}_{BB'|\{G_{AA'}^{(l)}\}}) - S(\tilde{\rho}_{BB'|AA'}) = \sum_{k=0}^{d-1} r_k \delta_{BB'|A}(\rho_k)$. By noticing that $\{G^{(l)}\}$ may not be the optimal POVM for the quantum discord of $\tilde{\rho}$, we have proven Eq. (33). Since Eq. (29) is a general form of the optimal input state, the above discussion shows that

$$\mathcal{Q}(\Lambda^{\rm MP} \otimes \Lambda^{\rm CD}) \leqslant \mathcal{Q}(\Lambda^{\rm MP}). \tag{35}$$

Combining Eq. (35) with Theorem 3, we finally reach Eq. (28). This completes the proof of Theorem 4.

It should be noticed that both the MP channel and the CD channel are coherence-breaking channels. A CD channel takes any state to a state which is diagonal on a fixed basis, and thus the coherence between different state bases is broken. For a MP channel, coherence is broken during the measurement and so is the genuine quantum correlation. Even through the preparation process in the MP channel can rebuild the bipartite quantum correlation, the genuine quantum correlation, which enables the superadditivity of QCP, cannot be rebuilt. This is the reason why the QCP of a MP channel and a CD channel is additive. Therefore, we conjecture that the QCP of channels which are neither MP channels nor CD channels is superadditive.

IV. CONCLUSION

We have investigated the effect of superadditivity of QCP, which means that using two channels together is more efficient in creating quantum correlations than using them separately. Two zero-QCP channels have the property of superactivation of QCP except the trivial cases where the two channels are both completely deochering channels or unitary channels. This result shows that superactivation of QCP is a fairly common effect for local channels. We also prove the superadditivity of QCP for general local channels and find a class of quantum channels whose QCP is additive.

Superadditivity of QCP is a collective effect. The genuine quantum correlation is observed in the initial state which can detect the superactivation of QCP for a CD channel in parallel with a depolarizing channel. Meanwhile, the QCP of a MP channel and a CD channel is additive since both of them are coherence-breaking channels, which break the genuine quantum correlation. Therefore, we conjecture that genuine quantum correlation is responsible for such an effect. This provides an alternate perspective from which to look at the concept of genuine quantum correlation, which is still an open problem in quantum-information theory. From this point of view, our study can shed light on both the classification of quantum channels and the structure of quantum correlation in multipartite states.

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